MATH-UA 325 Analysis I
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Series

Geometric Series

Properties of Series

Harmonic Series

p-Series

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Series

- Given a sequence $(x_n : \ n \geq n_0)$, consider the infinite sum
$$\sum_{n=n_0}^{\infty} x_n = x_{n_0} + x_{n_0+1} + \cdots ,$$
which is also called a series
- The sequence of partial sums is $(s_n : n \geq n_0)$, where
$$s_n = x_{n_0} + \cdots + x_n = \sum_{k=n_0}^{n} x_k$$
- The infinite sum is defined to be **convergent** if the sequence of partial sums is convergent
  - If so, we say that
$$\sum_{n=n_0}^{\infty} x_n = \lim_{n \to \infty} s_n$$
- If the sequence of partial sums diverges, then the series is **divergent**

Tails of Sequences and Series

- A **tail** of a sequence $(s_n : \ n \geq n_0)$ is a sequence

$$(s_n : n \geq N),$$

where N is some integer greater than n_0

  - A sequence converges if and only if a tail does

- A **tail** of a series

$$\sum_{n=n_0}^{\infty} x_n$$

is a series

$$\sum_{n=N}^{\infty} x_n,$$

where N is some integer greater than n_0

  - A series converges if and only if a tail does

Geometric Series and Sum

- Geometric series: Given $r \in \mathbb{R}$, consider the series

$$1 + r + r^2 + \cdots = \sum_{k=0}^{\infty} r^k$$

- Basic algebraic formula:

$$1 - r^{N+1} = (1-r)(1 + r + r^2 + \cdots + r^N) = (1-r)\sum_{k=0}^{N} r^k$$

- Geometric sum:

$$\sum_{k=0}^{N} r^k = 1 + r + r^2 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r}$$

- Therefore,

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \sum_{k=0}^{N} r^k = \lim_{N\to\infty} \frac{1 - r^{N+1}}{1 - r}$$

Convergence of Geometric Series

- If $|r| > 1$, then geometric series is unbounded
- If $|r| < 1$, then

  $$\lim_{n\to\infty} r^n = 0,$$

  and therefore

  $$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty} \sum_{k=0}^{N} r^k = \lim_{N\to\infty} \frac{1 - r^{N-n+1}}{1-r} = \frac{1}{1-r}$$

Series Converges $\implies$ Limit of Terms Equals 0

- Suppose the series $\sum_{k=0}^{\infty} x_k$ converges
- Therefore, the sequence $(s_n : n \geq 0$, where

$$s_n = \sum_{k=0}^{n} x_k,$$

is convergent and therefore a Cauchy sequence
- For any $\epsilon > 0$, there exists $N_\epsilon > 0$, such that for any $j, k > N_\epsilon$,

$$|s_j - s_k| < \epsilon$$

  - In particular, if $k = j - 1$, it follows that

$$|x_j| = |s_j - s_{j-1}| < \epsilon$$

- It follows that

$$\lim_{k \to \infty} x_k = 0$$

Converse is Not True

- Harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

- For each $n \geq 0$, let

$$t_n = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} = \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n-1} + \frac{1}{2^{n+1}}$$

- Observe that since there are 2^n decreasing terms

$$t_n > \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

and therefore

$$s_{2^{N+1}} = \sum_{n=0}^{n=N} t_n > \frac{N}{2}$$

- It follows that the sequence $(s_n : \ n \geq 1)$ is unbounded and therefore diverges

Basic Properties of Series

- If

$$\sum_{n=n_0}^{\infty} x_n \text{ converges,}$$

then, for any $c \in \mathbb{R}$,

$$\sum_{n=n_0}^{\infty} cx_n = c \sum_{n=n_0}^{\infty} x_n$$

- If the series

$$\sum_{n=n_0}^{\infty} x_n \text{ and } \sum_{n=n_0}^{\infty} y_n \text{ converge,}$$

then

$$\sum_{n=n_0}^{\infty} (x_n + y_n) = \sum_{n=n_0}^{\infty} x_n + \sum_{n=n_0}^{\infty} y_n$$

Multiplication and Division of Series

- Multiplication and division of series are more complicated, because

$$\sum_{n=n_0}^{\infty} x_n y_n \neq \left(\sum_{n=n_0}^{\infty} x_n\right)\left(\sum_{n=n_0}^{\infty} y_n\right)$$

$$\sum_{n=n_0}^{\infty} \frac{x_n}{y_n} \neq \frac{\sum_{n=n_0}^{\infty} x_n}{\sum_{n=n_0}^{\infty} y_n}$$

Cauchy Series

- A series is **Cauchy** if the sequence of partial sums is a Cauchy sequence
- A series is convergent if and only if it is Cauchy
- A series

$$\sum_{n=n_0}^{\infty} x_n$$

is Cauchy if for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for any $N_\epsilon < j \leq k$,

$$\left| \left(\sum_{i=n_0}^{k} x_i \right) - \left(\sum_{i=n_0}^{j} x_i \right) \right| < \epsilon,$$

i.e.,

$$\left| \sum_{j+1}^{k} x_i \right| < \epsilon,$$