MATH-UA 325 Analysis I
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Product of Absoultely Convergent Series
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Product of Absolutely Convergent Series

- The product of two absolutely convergent series

$$\sum_{m=1}^{\infty} a_m \text{ and } \sum_{n=1}^{N} b_n,$$

is equal to the absolutely convergent sequence

$$\sum_{k=1}^{\infty} \sum_{j=1}^{k} a_j b_{k+1-j}$$

Comparison Test

- If

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{N} b_n,$$

are series such that for each $n \in \mathbb{N}$, $0 \leq a_n \leq b_n$, then

  - If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$ and

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{N} b_n$$

  - If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$

Limit Comparison Test

- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{N} b_n$ be series where

$$\forall n \in \mathbb{N},\ a_n, b_n > 0$$

  and

$$0 < \lim_{n \to \infty} \frac{a_n}{b_n} < \infty$$

- Either both series converge or both series diverge
- Note that it suffices to prove

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{N} b_n \text{ converges}$$

Proof of Limit Comparison Test

- If $\sum_{n=1}^{\infty} b_n$ converges, then it is Cauchy
    - For any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\forall k > j > N_\epsilon,\ \sum_{n=j+1}^{k} b_n < \epsilon$$

- Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$
    - For any $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that

$$\forall n > M_\epsilon,\ L - \epsilon < \frac{a_n}{b_n} < L + \epsilon$$

    - Therefore, for all $n > M_1$, $a_n < (L+1)b_n$
- For any $\epsilon > 0$ and $k > j > \max(N_{\epsilon/(L+1)}, M_1)$,

$$\sum_{n=j+1}^{k} a_n < (L+1)\sum_{n=j+1}^{n} b_n < \epsilon$$

- It follows that $\sum_{n=1}^{\infty} a_n$ is Cauchy and converges

p-Series if $p \leq 1$

- Given $p \in \mathbb{Q}$, the p-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- If $p \leq 1$, then for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{1}{n^p} \geq \sum_{n=1}^{N} \frac{1}{n},$$

  which diverges

p-Series if $p > 1$

- For each $n \geq 0$, let

$$t_n = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k^p} = \frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + \cdots + \frac{1}{(2^{(n} + 2^n - 1)^p}$$

- Observe that since t_n has 2^n decreasing terms,

$$t_n \leq \frac{2^n}{(2^n)^p} = \left(\frac{1}{2^{(p-1)}}\right)^n$$

- If $p > 1$, then $\dfrac{1}{2^{p-1}} < 1$, and therefore

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \text{ converges}$$

- By the comparison test, $\displaystyle\sum_{n=0}^{\infty} t_n$ converges

p-Series if $p > 1$

- For any $N \in \mathbb{N}$, if $M \in \mathbb{N}$ satisfies $N \leq 2^{M+1} - 1$, then

$$\sum_{k=1}^{N} \frac{1}{k^p} \leq \sum_{k=1}^{2^{M+1}-1} \frac{1}{k^p} = \sum_{n=0}^{M} t_n < \sum_{n=0}^{\infty} t_n$$

- It follows that the sequence of partial sums

$$\left(\sum_{k=1}^{n} \frac{1}{k^p} : \; n \geq 1 \right)$$

is bounded, increasing, and therefore convergent

- Therefore, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges

Example of Limit Comparison Test Using p-Series

- Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2} - n + 1}$
- Use the limit comparison test for

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2} - n + 1} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

- $$\begin{aligned} \lim_{n\to\infty} \frac{\frac{1}{n^{3/2}-n+1}}{\frac{1}{n^{3/2}}} &= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2} - n + 1} \\ &= \lim_{n\to\infty} \frac{1}{1 - n^{-1/2} + n^{-3/2}} \\ &= 1 \end{aligned}$$

- Since the second series converges, so does the first series
- Therefore, the original series is absolutely convergent

Ratio Test for a Series

- Let $\sum_{n=n_0}^{\infty} x_n$ be a series such that
  - For all $n \geq n_0$, $x_n \neq 0$
- If there exists $N \geq n_0$ and $0 < \rho < 1$ such that for all $n \geq N$,

$$\frac{|x_{n+1}|}{|x_n|} \leq \rho,$$

  then the series converges absolutely
- If there exists $N \geq n_0$ and $\rho > 1$ such that for all $n \geq N$,

$$\frac{|x_{n+1}|}{|x_n|} \geq \rho,$$

  then the series diverges

Proof of Ratio Test for Divergence

- If for all $n \geq N$,

$$\frac{|x_{n+1}|}{|x_n|} \geq \rho > 1,$$

  then

$$|x_n| \geq \rho |x_{n-1}| \geq \cdots \geq \rho^{N-n} |x_N|$$

- This implies that the sequence $(x_n : \; n \geq n_0)$ is unbounded and therefore the series

$$\sum_{n=n_0}^{\infty} x_n$$

  is divergent

Proof of Ratio Test for Convergence

- If for all $n \leq N$,

$$\frac{|x_{n+1}|}{|x_n|} \leq \rho < 1,$$

  then

$$|x_n| \leq \rho |x_{n-1}| \leq \cdots \leq \rho^{N-n} |x_N|$$

- Since

$$\sum_{n=N}^{\infty} \rho^{N-n} |x_N| \text{ converges,}$$

  it follows by the comparison test that

$$\sum_{n=N}^{\infty} |x_n| \text{ converges}$$

- Therefore,

$$\sum_{n=n_0}^{\infty} |x_n| = |x_{n_0}| + \cdots + |x_{N-1}| + \sum_{n=N}^{\infty} |x_n|$$

  converges

Example (Part 1)

- Given $x \in \mathbb{R}$, consider the series

$$\sum_{n=1}^{\infty} n^{100} x^n$$

- If $x = 0$, then the series equals 0
- Let

$$s_n = n^{100} x^n$$

- If $x \neq 0$, then

$$\frac{|s_{n+1}|}{|s_n|} = \left| \frac{(n+1)^{100} x^{n+1}}{n^{100} x^n} \right| = \left(1 + \frac{1}{n}\right)^{100} |x|$$

Example (Part 2)

- Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{100}=1,$$

it follows that for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\forall n > N_\epsilon,\ 1-\epsilon < \left(1+\frac{1}{n}\right)^{100} < 1+\epsilon$$

- If $|x| < 1$, then for any $n > N_\epsilon$,

$$\left(1+\frac{1}{n}\right)^{100}|x| \leq (1+\epsilon)|x|$$

Example (Part 3)

- Choose $\epsilon > 0$ small enough so that

$$(1+\epsilon)|x| < 1,$$

- I.e., given ρ such that $0 < |x| < \rho < 1$, let

$$\epsilon = \frac{\rho}{|x|} - 1$$

- It follows that for all $n > N_\epsilon$,

$$\frac{|s_{n+1}|}{|s_n|} \leq \rho < 1$$

- By the ratio test, if $|x| < 1$, then

$$\sum_{n=1}^{\infty} n^{100} x^n = \sum_{n=1}^{\infty} s_n \text{ converges absolutely}$$

Splitting a Series into the Sum of Series

- A series

$$\sum_{n=1}^{\infty} x_n$$

can be split into the sum of two series as follows

- Let $S_1, S_2 \subset \mathbb{N}$ be disjoint subsets such that $S_1 \cup S_2 = \mathbb{N}$
- The series can be written as

$$\sum_{n=1}^{\infty} x_n = \left(\sum_{n \in S_1} x_n \right) + \left(\sum_{n \in S_2} x_n \right)$$

- The series on the left is absolutely convergent if and only if both series on the right are absolutely convergent