MATH-UA 325 Analysis I
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Power Series
Continuous Functions

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Power Series

- Given $x_0 \in \mathbb{R}$, a power series centered at x_0 is a series of the form
$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$
- For each $x \in \mathbb{R}$, this series either converges or diverges
- If $D \subset \mathbb{R}$ is given by
$$D = \left\{ x \in \mathbb{R} : \ \sum_{n=0}^{\infty} a_n(x - x_0)^n \text{ converges} \right\},$$
then the power series defines a function $f : D \to \mathbb{R}$, where, for each $x \in D$,
$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$
- Usually, we will restrict D to where the power series converges absolutely and define

Example: Geometric Power Series Centered at 0

- Recall that if $-1 < r < 1$

$$\sum_{n=0}^{\infty} r^n \text{ converges absolutely to } \frac{1}{1-r}$$

- It follows that the power series (centered at 0)

$$\sum_{n=0}^{\infty} x^n$$

is the function $f : (-1,1) \to \mathbb{R}$, given by

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

- Observe that the domain of f is $(-1,1)$, but the domain of the function

$$x \mapsto \frac{1}{1-x}$$

is $\mathbb{R}\setminus\{1\}$

Example: Geometric Power Series Centered at 1

- The geometric series centered at $x = 1$ is

$$\sum_{n=0}^{\infty}(x-1)^n,$$

  which converges absolutely if $-1 < x - 1 < 1$, i.e., $0 < x < 2$
- It is equal to the function $f : (0, 2) \to \mathbb{R}$ given by

$$f(x) = \frac{1}{1-(x-1)} = \frac{1}{2-x}$$

Example

- Consider

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- If

$$s_n = \frac{x^n}{n!},$$

then

$$\begin{aligned} \lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} &= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \frac{n!}{|x|^n} \\ &= \frac{|x|}{n+1} \\ &= 0 \end{aligned}$$

- By the ratio test, this power series converges absolutely for all $x \in \mathbb{R}$ and defines a function $f : \mathbb{R} \to \mathbb{R}$

Radius of Convergence

- Apply the ratio test to the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- If $s_n = a_n(x - x_0)^n$, then

$$\begin{aligned} \lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |x - x_0| \\ &= \frac{|x - x_0|}{R}, \end{aligned}$$

where

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

- If $x_0 - R < x < x_0 + R$, then the power series converges absolutely
- If $x < x_0 - R$ or $x > x_0 + R$, then the power series diverges

Discontinuity of a Function

- Given $S \subset \mathbb{R}$, consider a function $f : S \to \mathbb{R}$
- If there exists $x_0 \in S$ and a sequence $(x_n : \ n \geq 1) \subset S$ such that

$$\lim_{n\to\infty} x_n = x_0, \text{ but } \lim_{n\to\infty} f(x_n) \neq f(x_0),$$

  then f is **discontinuous** at x_0
- Or f has a **discontinuity** at x_0
- Example: The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R}, f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

  has a discontinuity at 0

Definition of a Continuous Function

- Given a $S \subset \mathbb{R}$, a function $f : S \to \mathbb{R}$ is **continuous** if it has no discontinuities
- I.e., for each $x_0 \in S$ and sequence

$$(x_n : \; n \geq 1) \text{ such that } \lim_{n \to \infty} x_n = x_0,$$

  the following holds:

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$

- Example: If $S = \{0, 1\}$, the function $f : S \to \mathbb{R}$ given by

$$f(0) = -5 \text{ and } f(1) = 17$$

  is continuous
- In practice, S will almost always be a nonempty open interval (a, b) or closed interval $[a, b]$, where $a < b$

Examples

- The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

is not continuous, because

$$\lim_{n\to\infty} \frac{1}{n} = 0, \text{ but } \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \neq 1 = f(0)$$

- On the other hand, the restriction of f to $\mathbb{R}\backslash\{0\}$,

$$f|_{\mathbb{R}\backslash\{0\}}(x) = 0, \ \forall x \in \mathbb{R}\backslash\{0\}$$

is continuous

Examples

- $\forall x \in \mathbb{R}$, $f(x) = 2x - 3$ is continuous
  - For each $x_0 \in \mathbb{R}$ and sequence $(x_n : \ n \geq 1)$ such that

$$\lim_{n \to \infty} x_n = x_0,$$

the following holds:

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 2x_n - 3 = 2(\lim_{n \to \infty} x_n) - 3 = 2x_0 - 3 = f(x_0)$$

- $\forall x \in \mathbb{R}$, $f(x) = x^2$ is continuous
  - For each $x_0 \in \mathbb{R}$ and sequence $(x_n : \ n \geq 1)$ such that

$$\lim_{n \to \infty} x_n = x_0,$$

the following holds:

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n^2 = (\lim_{n \to \infty} x_n)^2 = x_0^2 = f(x_0)$$