MATH-UA 325 Analysis I
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Continuous Functions
Limit of a Function

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Linear Combination of Continuous Functions is Continuous

- Given $S \subset \mathbb{R}$, reals $a, b \in \mathbb{R}$, and continuous functions $f, g : S \to \mathbb{R}$, let $h : S \to \mathbb{R}$ be given by

$$\forall x \in S,\ h(x) = af(x) + bg(x)$$

- Let $x_0 \in S$ and $(x_n :\ n \geq 1) \subset S$ be a sequence such that

$$\lim_{n\to\infty} x_n = x_0$$

- Since f and g are continuous,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \text{ and } \lim_{n\to\infty} g(x_n) = g(x_0)$$

- Therefore,

$$\begin{aligned}\lim_{n\to\infty} h(x_n) &= \lim_{n\to\infty} af(x_n) + bg(x_n) \\ &= a(\lim_{n\to\infty} f(x_n)) + b(\lim_{n\to\infty} g(x_n)) \\ &= af(x_0) + bg(x_0) = h(x_0)\end{aligned}$$

Product of Continuous Functions is Continuous

- Given $S \subset \mathbb{R}$ and continuous functions $f, g : S \to \mathbb{R}$, let $p : S \to \mathbb{R}$ be given by

$$\forall x \in S,\ p(x) = f(x)g(x)$$

- Let $x_0 \in S$ and $(x_n :\ n \geq 1) \subset S$ be a sequence such that

$$\lim_{n\to\infty} x_n = x_0$$

- Since f and g are continuous,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \text{ and } \lim_{n\to\infty} g(x_n) = g(x_0)$$

- Therefore,

$$\begin{aligned}\lim_{n\to\infty} h(x_n) &= \lim_{n\to\infty} f(x_n)g(x_n) \\ &= (\lim_{n\to\infty} f(x_n))(\lim_{n\to\infty} g(x_n)) \\ &= f(x_0)g(x_0) = h(x_0)\end{aligned}$$

Polynomials are Continuous

- Given $a_0, \ldots, a_d \in \mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R},\ f(x) = a_0 + a_1 x + \cdots + a_d x^d = \sum_{k=0}^{d} a_k x^k$$

  is continuous

  - For each $x_0 \in \mathbb{R}$, if $\lim_{n\to\infty} x_n = x_0$, then

$$\begin{aligned} \lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} \sum_{k=0}^{d} a_k x_n^k = \sum_{k=0}^{d} a_k (\lim_{n\to\infty} x_n^k) = \sum_{k=0}^{d} a_k (\lim_{n\to\infty} x_n)^k \\ &= \sum_{k=0}^{d} a_k x_0^k = f(x_0) \end{aligned}$$

Composition of Continuous Functions is Continuous

- Given subsets $S, T \subset \mathbb{R}$ and continuous functions $f : S \to T$, $g : T \to \mathbb{R}$, let $c = g \circ f : S \to \mathbb{R}$
- Let $x_0 \in S$ and $(x_n : \ n \geq 1) \subset S$ be a sequence such that

$$\lim_{n\to\infty} x_n = x_0$$

- Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x_0) \in T$
- Since g is continuous,

$$\begin{aligned}
\lim_{n\to\infty}(g \circ f)(x_n) &= \lim_{n\to\infty}(g \circ f)(x_n) \\
&= \lim_{n\to\infty} g(f(x_n)) \\
&= g(\lim_{n\to\infty} f(x_n)) \\
&= g(f(x_0)) \\
&= c(x_0)
\end{aligned}$$

Limit of Function

- Given a set $S \subset \mathbb{R}$, a function $f : S \to \mathbb{R}$, and $x_0 \in \mathbb{R}$,

$$\lim_{x \to x_0} f(x) = L,$$

  if the following holds:

  - If $(x_n : \ n \geq 1) \subset S$ is a Cauchy sequence such that

$$x_0 \notin (x_n : \ n \geq 1) \text{ and } \lim_{n \to \infty} x_n = x_0,$$

    then

$$\lim_{n \to \infty} f(x_n) = L$$

- Important Features of a Limit
  - x_0 need not be in S
  - L need not be equal to $f(x_0)$

Equivalent Definition of Limit

- ,

$$\lim_{x \to x_0} f(x) = L,$$

if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in S$,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

Function Continuous at a Point

- Given $S \subset \mathbb{R}$, consider a function $f : S \to \mathbb{R}$
- f is **continuous at** $x_0 \in S$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- f is continuous if it is continuous at every $x \in S$

Example

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$
- If $\lim_{n\to\infty} x_n = 1$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 = (\lim_{n\to\infty} x_n)^2 = 1$$

- If $\delta > 0$ and $|x-1| < \delta$, then

$$|f(x)-1| = |x^2-1| = |(x+1)(x-1)| = |x+1||x-1| < |x+1|\delta$$

- So if $0 < \delta < 1$, then if $|x-1| < \delta$, then

$$1-\delta < x < 1+\delta, \text{ which implies } x+1 < 2+\delta < 3$$

- It follows that if $0 < \delta < 1$, then

$$|f(x)-1| < 3\delta$$

- Set $\delta = \dfrac{\epsilon}{3}$