MATH-UA 325 Analysis I
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Compact Subsets of $\mathbb{R}$

Deane Yang

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Courant Institute of Mathematical Sciences
New York University

Compact Subsets of $\mathbb{R}$

- A set $S \subset \mathbb{R}$ is **(sequentially) compact** if any sequence in S has a convergent subsequence whose limit is in S
- **Theorem:** A set $S \subset \mathbb{R}$ is compact if and only if it is closed and bounded
- Closed and bounded $\implies$ compact
    - Any sequence in a bounded set S is bounded
    - By Bolzano-Weierstrass Theorem, it has a convergent subsequence
    - Since S is closed, the limit is also in S

Compact $\implies$ Closed and Bounded

- Compact $\implies$ closed
  - Given any Cauchy sequence $(x_n : n \geq 1) \subset S$, it has a convergent subsequence with limit $L \in S$
  - Therefore, $\lim_{n\to\infty} x_n = L \in S$
  - Since this holds for any Cauchy sequence in S, it follows that S is closed
- Compact $\implies$ bounded
  - Contrapositive: Unbounded $\implies$ not compact
  - If S is unbounded, then there exists a sequence $(x_n : n \geq 1) \subset S$ such that for each $n \geq 1$, $|x_n| > n$
  - Such a sequence has no convergent subsequence

Compact Set Contains Its Supremum and Infimum

- Let $C \subset \mathbb{R}$ be compact
- Since C is bounded, there exist

$$b = \inf C \text{ and } B = \sup C$$

- There exist sequences $(s_n : \; n \geq 1), (t_n : \; n \geq 1) \subset C$ such that

$$\lim_{n \to \infty} s_n = b \text{ and } \lim_{n \to \infty} t_n = B$$

- Since C is closed, $b, B \in C$