MATH-UA 325 Analysis I
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Intermediate Value Theorem

Deane Yang

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Courant Institute of Mathematical Sciences
New York University

Intermediate Value Theorem (Special Case)

Lemma
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof of Lemma

- Define sequences $(a_n : \ n \geq 0)$ and $(b_n : \ n \geq 0)$ by induction
- Let $a_0 = a$ and $b_0 = b$
- Given (a_n, b_n), define (a_{n+1}, b_{n+1}) as follows:
- Let $c_n = \dfrac{a_n + b_n}{2} f$
    - If $f(c_n) < 0$, let $a_{n+1} = a_n$ and $b_{n+1} = c_n$
    - If $f(c_n) > 0$, let $a_{n+1} = c_n$ and $b_{n+1} = b_n$
- The sequence $(a_n : \ n \geq 0)$ is increasing and bounded from above by b
    - Therefore, it is convergent
- The sequence $(b_n : \ n \geq 0)$ is decreasing and bounded from below by a
    - Therefore, it is convergent

Proof, Continued

- For each n, $b_n - a_n = \dfrac{b-a}{2^n}$
- Therefore,

$$\begin{aligned}(\lim_{n\to\infty} b_n) - (\lim_{n\to\infty} a_n) &= \lim_{n\to\infty}(b_n - a_n)\\ &= 0\end{aligned}$$

- Denote

$$c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof, Continued

- Since f is continuous

$$\lim_{n\to\infty} f(a_n) = f(a_\infty) = f(c)$$
$$\lim_{n\to\infty} f(b_n) = f(b_\infty) = f(c)$$

- Since $\forall n \geq 0, f(a_n) \leq 0$,

$$f(c) = \lim_{n\to\infty} f(a_n) \leq 0$$

  and, since $\forall n \geq 0, f(b_n) \geq 0$,

$$f(c) = \lim_{n\to\infty} f(b_n) \geq 0$$

- It follows that $f(c) = 0$

Intermediate Value Theorem

Theorem
Let $f : [a, b] \to \mathbb{R}$ *be a continuous function such that* $f(a) \neq f(b)$*. For any* y *between* $f(a)$ *and* $f(b)$*, there exists* $x \in [a, b]$ *such that* $f(x) = y$

Proof of Intermediate Value Theorem

- Let $h : [a, b] \to \mathbb{R}$ by given by

$$h(x) = f(x) - y$$

- If $f(a) < y < f(b)$, then $h(a) = f(a) - y < 0$ and $h(b) = f(b) - y > 0$
  - By lemma, there exists $x \in (a, b)$ such that $h(x) = 0$ and therefore $f(x) = y$
- If $f(a) > y > f(b)$, then $h(a) = f(a) - y > 0$ and $h(b) = f(b) - y > 0$
  - By lemma, there exists $x \in (a, b)$ such that $h(x) = 0$ and therefore $f(x) = y$

Any Polynomial of Odd Degree Has At Least 1 Root

- Given any odd $d \in \mathbb{N}$ and $a_0, \dots, a_{d-1} \in \mathbb{R}$, let

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0,$$

- Observe that

$$p(x) = x^d \left(1 + \frac{a_{d-1}}{x} + \cdots + \frac{a_0}{x^d}\right)$$

- If $|x| > 1$, then for any $p \geq 1$,

$$\frac{1}{|x|^p} < \frac{1}{|x|}$$

- Therefore, if $|x| > 1$ and $b = \max(|a_0|, \dots, |a_{d-1}|)$, then

$$\begin{aligned}\left|\frac{a_{d-1}}{x} + \cdots + \frac{a_0}{x^d}\right| &\leq \frac{|a_{d-1}|}{|x|} + \cdots + \frac{|a_0|}{|x^d|} \\ &\leq \frac{bd}{|x|},\end{aligned}$$

Any Polynomial of Odd Degree Has At Least 1 Root

- It follows that if $|x| > 1$, then

$$1 - \frac{bd}{|x|} < 1 + \frac{a_{d-1}}{x} + \cdots + \frac{a_0}{x^d} < 1 + \frac{bd}{|x|}$$

- So if $|x| > bd$, then

$$0 < 1 - \frac{bd}{|x|} < 1 + \frac{a_{d-1}}{x} + \cdots + \frac{a_0}{x^d}$$

- Therefore, if $t > bd$, then $(-t)^d < 0$ and

$$p(-t) = t^d \left(1 + \frac{a_{d-1}}{t} + \cdots + \frac{a_0}{t^d}\right) < 0$$

and

$$p(t) = t^d \left(1 + \frac{a_{d-1}}{t} + \cdots + \frac{a_0}{t^d}\right) > 0$$

- By the Special Case of Intermediate Value Theorem, there exists $c \in (-t, t)$ such that

$$p(c) = 0.$$