MATH-UA 325 Analysis I
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Properties of Continuous Functions

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Images of Open, Closed, Bounded Sets by Continuous Function

- Let I be an open interval and $f : I \to \mathbb{R}$ be continuous
- If $S \subset I$ is open, then $f(S)$ need not be open
    - Let $f : \mathbb{R} \to \mathbb{R}$ be a constant function and $S \subset \mathbb{R}$ be open
    - Let $f(x) = x^2$ for all $x \in \mathbb{R}$ and $S = (-1, 1)$
- If $S \subset I$ is closed, then $f(S)$ need not be closed
    - Let $f(x) = \dfrac{1}{x}$ for all $x > 0$ and $S = [1, \infty)$
- If $S \subset I$ is bounded, then $f(S)$ need not be bounded
    - Let $f(x) = \dfrac{1}{x}$ and $S = (0, 1)$

Image of Compact Set by Continuous Function is Compact

- Let $C \subset \mathbb{R}$ be compact and $f : C \to \mathbb{R}$ be continuous
- Let $(y_n \ : \ n \geq 1) \subset f(C)$
- Since each $y_n \in f(C)$, there exists $x_n \in C$ such that $f(x_n) = y_n$
- Since $(x_n : \ n \geq 1) \subset C$ and C is compact, there exists a subsequence $(x_{n_k} : \ k \geq 1)$ converging to a limit $x_0 \in C$
- Since f is continuous,

$$\lim_{k \to \infty} f(x_{n_k}) = f(x_0) \in f(C)$$

- Therefore, the subsequence $(y_{n_k} : \ k \geq 1) \subset f(C)$ converges to $y_0 = f(x_0) \in f(C)$
- Therefore, any sequence $(y_n : \ n \geq 1) \subset f(C)$ has a convergent subsequence whose limit is in $f(C)$
- It follows that $f(C)$ is compact

Two Definitions of a Continuous Function

- $f : S \to \mathbb{R}$ is continuous if for any $x_0 \in S$ and sequence $(x_n : \ n \geq 1) \subset S$ whose limit is x_0, the sequence $(f(x_n) : \ n \geq 1)$ is convergent and converges to $f(x_0)$
- If $I \subset \mathbb{R}$ is open, then $f : I \to \mathbb{R}$ is continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon,$$

i.e.,

$$x \in (x - \delta, x + \delta) \implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$$

Inverse Image of Open Set by Continuous Function

- Let $I \subset \mathbb{R}$ be open and $f : I \to \mathbb{R}$ be continuous
- If $T \subset \mathbb{R}$ is open, then for any $y_0 \in T$, there exists $\epsilon > 0$ such that

$$(y_0 - \epsilon, y_0 + \epsilon) \subset T$$

- Since f is continuous, there exists $\delta > 0$ such that

$$x \in (x_0 - \delta, x_0 + \delta) \implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$$

- Therefore, if $x_0 \in f^{-1}(T)$, then

$$\begin{aligned} x \in (x_0 - \delta, x_0 + \delta) &\implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon) \\ &\implies f(x) \in T \implies x \in f^{-1}(T) \end{aligned}$$

- Therefore, for any $x_0 \in f^{-1}(T)$, there exists $\delta > 0$ such that

$$(x_0 - \delta, x + 0 + \delta) \subset f^{-1}(T)$$

- Therefore, $f^{-1}(T)$ is open

Inverse Image of Open is Open $\implies$ Continuous

- Let I be open and $f : I \to \mathbb{R}$ be a function such that

$$O \subset \mathbb{R} \text{ open} \implies f^{-1}(O) \text{ open}$$

- Let $x_0 \in I$
- For any $\epsilon > 0$, $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ is open
- By definition of inverse image, $x_0 \in f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$
- By assumption, $f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$ is open
- Therefore, there exists $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon))$$

- By definition of inverse image, it follows that

$$x \in (x_0 - \delta, x_0 + \delta) \implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$$

- Therefore, f is continuous at each $x_0 \in I$

Inverse Image of Closed Set by Continuous Function is Closed

- Let I be an open interval and $f : I \to \mathbb{R}$ be continuous
- If $S \subset I$ is closed, then $I \backslash S$ is open
- If f is continuous, $f^{-1}(I \backslash S)$ is open
- $f^{-1}(I \backslash S) \cap f^{-1}(S) = \emptyset$
  - If $x \in f^{-1}(S)$, then $f(x) \in S$ and therefore, $f(x) \notin I \backslash S$
- $f^{-1}(I \backslash S) \cup f^{-1}(S) = I$
- Therefore, $I \backslash (f^{-1}(I \backslash S)) = f^{-1}(S)$
- Therefore,

$$\begin{aligned} S \subset I \text{ closed } &\implies I \backslash S \text{ open} \\ &\implies f^{-1}(I \backslash S) \text{ open} \\ &\implies f^{-1}(S) \text{ close} \end{aligned}$$