MATH-UA 325 Analysis I
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Extreme Value Theorem
Derivative of Function
Product Rule

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Extreme Value Theorem

- If $S \subset \mathbb{R}$ is compact and $f : S \to \mathbb{R}$ is continuous, then there exists $x_{\min}, x_{\max} \in S$ such that for any $x \in S$,

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

- Proof
    - If S is compact, then $f(S)$ is compact
    - If $f(S)$ is compact, then

$$y_{\min} = \inf f(S), y_{\max} = \sup f(S) \in f(S)$$

    - Since $y_{\min}, y_{\max} \in f(S)$, there exist $x_{\min}, x_{\max}$ such that

$$f(x_{\min}) = y_{\min} \text{ and } f(x_{\max}) = y_{\max}$$

Derivative of a Function

- Let $I \subset \mathbb{R}$ be open and $f : I \to \mathbb{R}$ be a function
- Given any $x_0 \in I$, if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

  exists, then f is **differentiable** at x_0
- The limit is called the **derivative** of f at x_0 and denoted

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- If f is differentiable at every $x_0 \in I$, then its derivative is a function

$$\begin{aligned} f' : I &\to \mathbb{R} \\ x_0 &\mapsto \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \end{aligned}$$

Differentiable $\implies$ Continuous

- Suppose $f : I \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$
- If $m = f'(x_0)$, then

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Let $(x_n : \; n \geq 1) \subset I$ be any sequence such that

$$\forall n \geq 1, x_n \neq x_0 \text{ and } \lim_{n \to \infty} x_n = x_0$$

- Since the sequences

$$\left(\frac{f(x_n) - f(x_0)}{x_n - x_0} : \; n \geq 1 \right) \text{ and } (x_n - x_0 : \; n \geq 1)$$

both converge, so does their product

$$\begin{aligned} \lim_{n \to \infty} (f(x_n) - f(x_0)) &= \lim_{n \to \infty} \left(\frac{f(x_n) - f(x_0)}{x_n - x_0} \right) (x_n - x_0) \\ &= \left(\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \right) \left(\lim_{n \to \infty} x_n - x_0 \right) \end{aligned}$$

Differentiable $\implies$ Continuous

- If $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then for any sequence $(x_n : \; n \geq 1)$ such that

$$\forall n \geq 1, x_n \neq x_0 \text{ and } \lim_{n\to\infty} x_n = x_0,$$

  it follows that

$$\lim_{n\to\infty} f(x) = f(x_0)$$

- Therefore, f is continuous at x_0

Linear Approximation of Differentiable Function

- $f'(x_0) = \lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$
- Equivalently,

$$\lim_{x \to x_0} \frac{f(x) - (y_0 + m(x - x_0))}{x - x_0} = 0$$

where $y_0 = f(x_0)$ and $m = f'(x_0)$

- $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ if there exists $y_0, m \in \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - (y_0 + m(x - x_0))}{x - x_0} = 0$$

- If so, $f(x_0) = y_0$ and $f'(x_0) = m$

Rules of Differentiation

- Let $I \subset \mathbb{R}$ be open and $f, f_1, f_2 : I \to \mathbb{R}$ be differentiable
- Easy rules
  - If f is a constant function, then f' is the zero function
  - If $\forall x \in \mathbb{R},\ f(x) = x$, then f' is the constant function equal to 1
  - (Constant factor rule) If $c \in \mathbb{R}$, then $(cf)' = cf'$
  - $(f_1 + f_2)' = f_1' + f_2'$
- Less easy rules
  - (Product rule) $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$
  - (Reciprocal rule) $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$
  - (Quotient rule) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

Product Rule (Part 1)

- Let $f, g : I \to \mathbb{R}$ be differentiable at $x_0 \in I$
- Linear approximations

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$$
$$g(x) \simeq g(x_0) + g'(x_0)(x - x_0)$$

- Therefore, we believe that

$$\begin{aligned} f(x)g(x) &\simeq (f(x_0) + f'(x_0)(x - x_0))(g(x_0) + g'(x_0)(x - x_0)) \\ &= f(x_0)g(x_0) + (f'(x_0)g(x_0) + f(x_0)g'(x_0))(x - x_0) \\ &\quad + f'(x_0)g'(x_0)(x - x_0)^2 \end{aligned}$$

- This suggests that

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Product Rule (Part 2)

- The difference quotient for fg is

$$
\begin{aligned}
&\frac{(fg)(x) - (fg)(x_0)}{x - x_0} \\
&= \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{(f(x) - f(x_0))g(x) + f(x_0)(g(x) - g(x_0))}{x - x_0} \\
&= g(x)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) + f(x_0)\left(\frac{g(x) - g(x_0)}{x - x_0}\right)
\end{aligned}
$$

Product Rule (Part 3)

- Therefore, if f and g are differentiable at x_0,

$$\begin{aligned}\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= (\lim_{x \to x_0} g(x)) \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \\ &\quad + f(x_0) \left(\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \right) \\ &= g(x_0) f'(x_0) + f(x_0) g'(x_0)\end{aligned}$$