MATH-UA 325 Analysis I
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Mean Value Theorem

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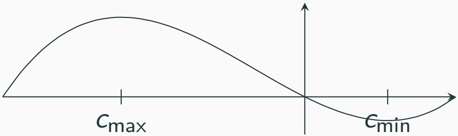
Relative Maxima and Minima



- Let $I \subset \mathbb{R}$ be open and let $f : I \to \mathbb{R}$ be a function
- f has a **relative maximum** at $c \in I$ if there exists $\delta > 0$ such that for any $x \in (c - \delta, c + \delta)$,

$$f(x) \leq f(c)$$

- f has a **relative minimum** at $c \in I$ if there exists $\delta > 0$ such that for any $x \in (c - \delta, c + \delta)$,

$$f(x) \geq f(c)$$

- f has a **relative extremum** at $c \in I$ if c is a relative maximum or relative minimum of f

Relative Extremum is Critical Point (Part 1)

- Let $I \subset \mathbb{R}$ be open and let $f : I \to \mathbb{R}$ be a differentiable function
- Let $c \in I$ be a relative maximum of f
- There exists $\delta > 0$ such that for any $x \in (c - \delta, c + \delta)$,

$$f(x) \leq f(c), i.e., f(x) - f(c) \leq 0$$

- Therefore,

$$\frac{f(x) - f(c)}{x - c} \begin{cases} \geq 0 & \text{if } x < c \\ \leq 0 & \text{if } x > c \end{cases}$$

Relative Extremum is Critical Point (Part 2)

- If $(x_n : \ n \geq 1)$ is a sequence such that

$$\forall n \geq 1, \ x_n < c,$$

then

$$f'(c) = \lim_{n \to \infty} \frac{f(x) - f(c)}{x - c} \geq 0$$

- If $(x_n : \ n \geq 1)$ is a sequence such that

$$\forall n \geq 1, \ x_n > c,$$

then

$$f'(c) = \lim_{n \to \infty} \frac{f(x) - f(c)}{x - c} \leq 0$$

- Therefore, $f'(c) = 0$
- $c \in I$ is a critical point of a function $f : I \to \mathbb{R}$ if either f is not differentiable at c or $f'(c) = 0$

Rolle's Theorem

- **Theorem:** If $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$
- **Proof**
  - Since $[a, b]$ is compact, there exists $c_{\min}, c_{\max} \in [a, b]$ such that

    $$\forall x \in [a, b],\ f(c_{\min}) \leq f(x) \leq f(c_{\max})$$

  - If f is constant, then $f'(c) = 0$ for every $c \in (a, b)$
  - Otherwise, there exists $x \in (a, b)$ such that

    $$f(x) \neq f(a) = f(b)$$

  - If $f(x) > f(a) = f(b)$, then

    $$f(c_{\max}) \geq f(x) > f(a) = f(b)$$

    and therefore $c_{\max} \in (a, b)$
  - Since $c_{\max}$ is a relative maximum, $f'(c_{\max}) = 0$
  - If $f(x) < f(a) = f(b)$, then $c_{\min} \in (a, b)$ and $f'(c_{\min}) = 0$

Mean Value Theorem

- **Theorem:** If $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Proof**

Define $g : [a, b] \to \mathbb{R}$ by

$$\forall x \in [a, b],\ g(x) = f(x) - \left(f(a) + \left(\frac{f(b) - f(a)}{b - a} \right) (x - a) \right)$$

  - g is continuous on $[a, b]$ and differentiable on (a, b)
  - $g(a) = g(b) = 0$
  - By Rolle's Theorem, there exists $c \in (a, b)$ such that

$$0 = g'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b - a} \right)$$