MATH-UA 325 Analysis I
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Applications of Mean Value Theorem
Taylor Polynomials
Taylor's Theorem

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Applications of Mean Value Theorem

- Let $f : (a, b) \to \mathbb{R}$ be differentiable
  - $f' = 0$ on (a, b) if and only if f is a constant function
  - f is increasing on $(a, b) \iff f' \geq 0$ on (a, b)
  - f is decreasing on $(a, b) \iff f' \leq 0$ on (a, b)
  - $f' > 0$ on $(a, b) \implies f$ is strictly increasing on (a, b)
  - $f' < 0$ on $(a, b) \implies f$ is strictly decreasing on (a, b)
  - If $c \in (a, b)$, $f' \geq 0$ on (a, c) and $f' \leq 0$ on (c, b), then c is an absolute maximum on (a, b)
  - If $c \in (a, b)$, $f' \leq 0$ on (a, c) and $f' \geq 0$ on (c, b), then c is an absolute minimum on (a, b)

Higher Order Derivatives

- Given an open $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$, denote $f^{(0)} = f$ and for each $k \geq 1$,
$$f^{(k)} = (f^{(k-1)})'$$
- Definition: f is C^k if $f^{(0)}, \ldots, f^{(k)}$ are continuous

Taylor Polynomials

- If $f : I \to \mathbb{R}$ is kth order differentiable, its Taylor polynomial of order k centered at $x_0 \in I$ is the polynomial of degree k,
$$P_k(x) = a_0(x - x_0) + a_1(x - x_0) + \cdots + a_k(x - x_0)^k$$
such that for each $0 \leq j \leq k$,
$$f^{(j)}(x_0) = P_k^{(j)}(x_0)$$
- A straightforward calculation shows that
$$a_j = \frac{f^{(j)}(x_0)}{j!}$$
and therefore
$$\begin{aligned} P_k(x) &= f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \end{aligned}$$

Little o Notation

- Given $x_0 \in \mathbb{R}$ and $\delta > 0$, let

$$e : (x_0 - \delta, x_0 + \delta) \to [0, \infty), \; g : (x_0 - \delta, x_0 + \delta) \to (0, \infty)$$

be continuous functions

- We write

$$e(x) = o(g(x)) \text{ as } x \to x_0 \text{ if } \lim_{x \to x_0} \frac{e(x)}{g(x)} = 0$$

- It implies that, as $x \to x_0$, $f(x)$ is approaching 0 faster than $g(x)$
- Given continuous functions $f_1, f_2 : (x_0 - \delta, x_0 + \delta) \to [0, \infty)$,

$$f_2(x) = f_1(x) + o(g(x)) \text{ as } x \to x_0$$

means

$$|f_2(x) - f_1(x)| = o(g(x)) \text{ as } x \to x_0, \text{ i.e., } \lim_{x \to x_0} \frac{|f_2(x) - f_1(x)|}{g(x)} = 0$$

Taylor's Theorem

If $f : (a, b) \to \mathbb{R}$ is k-th order differentiable and P_k is its k-th order Taylor polynomial centered at $x_0 \in (a, b)$, then

$$f(x) = P_k(x) + o(|x - x_0|^k) \text{ as } x \to x_0,$$

i.e.,

$$\lim_{x \to x_0} \frac{f(x) - P_k(x)|}{|x - x_0|^k} = 0$$

Taylor's Theorem for $k = 0$

- If $k = 0$, then $P_0(x) = f(x_0)$ and, since

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0)|}{1} = 0,$$

  it follows that

$$f(x) = f(x_0) + o(1)$$

Taylor's Theorem for $k = 1$

- If $k = 1$, then if

$$R_1(x) = f(x) - P_1(x) = f(x) - f(x_0) - f'(x_0)(x - x_0),$$

it follows by the Mean Value Theorem, there exists c_x between x_0 and x such that

$$\begin{aligned} R_1(x) - R_1(x_0) &= R'(c_x)(x - x_0) \\ &= (f'(c_x) - f'(x_0))(x - x_0) \end{aligned}$$

- Since $R_1(x_0) = 0$ and f' is continuous, it follows that

$$\begin{aligned} \lim_{x \to x_0} \frac{|f(x) - P_1(x)|}{|x - x_0|} &= \lim_{x \to x_0} \frac{|R_1(x) - R_1(x_0)|}{|x - x_0|} \\ &= \lim_{x \to x_0} |f'(c_x) - f'(x_0)| \\ &= 0 \end{aligned}$$

Taylor's Theorem for $k = 2$ (Part 1)

- Let

$$\begin{aligned} R_2(x) &= f(x) - P_2(x) \\ &= f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2} f''(x_0)(x - x_0)^2 \end{aligned}$$

- Observe that

$$R_2'(x) = f'(x) - f'(x_0) - f''(x_0)(x - x_0),$$

and $R_2(x_0) = R_2'(x_0) = 0$

- By the Mean Value Theorem twice, there exists c_x between x and $x_0$such that

$$\begin{aligned} f(x) - P_2(x) &= R_2(x) - R_2(x_0) \\ &= R_2'(c_x)(x - x_0) \end{aligned}$$

Taylor's Theorem for $k = 2$ (Part 2)

- Since $R_2'(x_0) = 0$, it follows the Mean Value Theorem that there exists d_x between c_x and x_0 such that

$$\begin{aligned} R_2'(c_x) &= R_2'(c_x) - R_2'(x_0) \\ &= R_2''(d_x)(x - x_0) \\ &= (f''(d_x) - f''(x_0))(x - x_0) \end{aligned}$$

- Therefore,

$$\begin{aligned} f(x) - P_2(x) &= R_2'(c_x)(x - x_0) \\ &= (f''(d_x) - f''(x_0))(x - x_0)^2 \end{aligned}$$

- Since f'' is continuous,

$$\lim_{x \to x_0} \frac{|f(x) - P_2(x)|}{|x - x_0|^2} = \lim_{x \to x_0} |f''(d_x) - f''(x_0)| = 0$$

- Therefore, $f(x) = P_2(x) + o(|x - x_0|^2)$