MATH-UA 325 Analysis I
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Derivative of Inverse Function
Integration

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Example: Inverse of Linear Function

- Given $a \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ be given by

$$\forall x \in \mathbb{R}, f(x) = ax + b,$$

  - f is bijective
  - Its inverse function is

$$f^{-1}(y) = \frac{y - b}{a}$$

  - Since $f'(x) = a$,

$$(f^{-1})'(y) = \frac{1}{a} = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}$$

Example: Inverse of Quadratic Function

- Let $g : (0, \infty) \to (0, \infty)$ be given by

$$\forall x \in (0, \infty),\ g(x) = x^2$$

  - g is bijective
  - Its inverse is

$$g^{-1}(y) = \sqrt{y}$$

  - The derivative of the inverse function is

$$(g^{-1})'(y) = \frac{1}{2\sqrt{y}} = \frac{1}{2x} = \frac{1}{g'(x)} = \frac{1}{g'(g^{-1}(y))}$$

Inverse Function

- Let $I, J \subset \mathbb{R}$
- Let $f : I \to J$ be a bijective function
- There is a uniquely defined function $g : J \to I$ that is bijective and satisfies

$$\forall y \in J,\ f(g(y)) = y \text{ and } \forall x \in I,\ g(f(x)) = x$$

- g is called the **inverse function** of f and usually denoted f^{-1}
- If I, J are intervals and $f : I \to J$ is continuous, then f is bijective if and only if f is strictly monotone

Derivative of an Inverse Function

- Let $I, J \subset \mathbb{R}$ be open
- Let $f : I \to J$ be bijective and $g : J \to I$ be its inverse
- If f and g are differentiable, then

$$f(g(y)) = y,$$

- By the chain rule, for all $y \in J$,

$$f'(g(y))g'(y) = 1$$

- This implies that for all $x \in I$, $f'(x) \neq 0$ and

$$\forall y \in J,\ g'(y) = \frac{1}{f'(g(y))},$$

  i.e.,

$$\forall y \in J,\ (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Inverse Function Theorem

Theorem
If $f : (a, b) \to \mathbb{R}$ is a differentiable function such that

$$\forall x \in (a, b),\ f'(x) \neq 0,$$

then

- *f is strictly monotone and therefore has an inverse function*

$$f^{-1} : f((a, b)) \to (a, b)$$

- *f^{-1} is differentiable and*

$$\forall y \in f((a, b)),\ (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

Linear Approximation of a Function

- Recall that if $f : (a, b) \to \mathbb{R}$ is C^1 and $x_0, x_1 \in (a, b)$, then by Taylor's theorem

$$f(x_1) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|),$$

i.e.,

$$f(x_1) - f(x_0) = f'(x_0)(x - x_0) + o(|x - x_0|),$$

- If $y_0 = f(x_0)$ and

$$y_1 = y_0 + f'(x_0)(x - x_0),$$

then y_1 is a linear approximation of $f(x_1)$

Reconstruction of a Function From Its Derivative (Part 1)

- Goal: Approximate $f(x)$ using only $f(x_0)$ and f'
- Key idea:
  - Chop the interval $[x_0, x]$ into small subintervals

    $$[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N], \text{ where } x_N = x$$

  - Use the linear approximation of f on each subinterval
- Given $y_0 = f(x_0)$ and the function f', let

$$\begin{aligned} y_1 &= y_0 + f'(x_0)(x_1 - x_0) \\ y_2 &= y_1 + f'(x_1)(x_2 - x_1) \\ &\vdots \\ y_N &= y_{N-1} + f'(x_N)(x_N - x_{N-1}) \end{aligned}$$

- Each y_k is an approximation of $f(x_k)$
- y_N is an approximation of $f(x)$

Example: Calculate Position Function from Velocity Function

- Let $\mathbb{R}$ represent a straight road going east-west, and $0 \in \mathbb{R}$ be a staring point
- Each $x \in \mathbb{R}$ represents a location on the road
- Suppose a car drives on the road (not always in the same direction) for a period of T hours
- For each time $t \in [0, T]$, let $p(t) \in \mathbb{R}$ be the position of the car at time t
- The (instantaneous) velocity function of the car is $v = p'$
- At each time t, $v(t)$ is known by the direction of travel and the speedometer
- We want to figure out, for each time t, the position $p(t)$ of the car
- Idea: Estimate distance and direction traveled for a consecutive sequence of small time intervals

Reconstruction of Function From Its Derivative (Part 2)

- The equations above can be written as:

$$
\begin{aligned}
y_1 - y_0 &= f'(x_0)(x_1 - x_0) \\
y_2 - y_1 &= f'(x_1)(x_2 - x_1) \\
\vdots \quad &\quad \vdots \\
y_N - y_{N-1} &= f'(x_{N-1})(x_N - x_{N-1})
\end{aligned}
$$

- Adding them all, we get

$$
f(x) - f(x_0) \simeq \sum_{k=1}^{N} f'(x_{k-1})(x_k - x_{k-1})
$$

- To get an exact answer, take a limit, where $N \to \infty$ and the size of each interval converges to zero,

$$
|x_k - x_{k-1}| \to 0
$$

Integral of a Function

- Consider a function $v : [0, T] \to \mathbb{R}$
- Given $N \in \mathbb{N}$, chop $[0, T]$ into equal sized subintervals:
  - Let $\delta = \frac{T}{N}$
  - $T_0 = 0,\ T_1 = \delta, T_2 = 2\delta, \ldots, T_N = N\delta$
- For each $k \in \{1, 2, \ldots, N\}$, let

$$d_k = v(T_{k-1})(T_k - T_{k-1}) = v(T_{k-1})\frac{T}{N}$$

- Let

$$S_N = d_1 + \cdots + d_N = \sum_{k=1}^{N} v(T_{k-1})(T_k - T_{k-1}) = \frac{T}{N}\sum_{k=1}^{N} v(T_{k-1})$$

- The **integral** of v over the interval $[0, T]$ is defined to be

$$\int_{t=0}^{t=T} v(t)\,dt = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{T}{N}\sum_{k=1}^{N} v\left(\frac{T}{N}\right),$$