MATH-UA 325 Analysis I
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Riemann Sum
Riemann Integral

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Partition and Subordinate Sequence

- A **partition** of a compact interval $[a, b]$ consists of a finite strictly monotone sequence

$$P = (x_0, x_1, \ldots, x_N)$$

  such that $x_0 = a$ and $x_N = b$
- In other words,

$$x_0 = a < x_1 < \cdots < x_N = b$$

- A monotone sequence $C = (c_1, \ldots, c_N)$ is **subordinate** to P if

$$\forall k \in \{1, \ldots, N\},\ c_k \in [x_{k-1}, x_k]$$

Riemann Sum

- Given a function $f : [a, b] \to \mathbb{R}$, a **Riemann sum** of f with respect to a partition P and subordinate sequence C is given by

$$S_f(P, C) = \sum_{k=1}^{N} f(c_k)(x_k - x_{k-1})$$

Riemann Sum Estimates Big Change as Sum of Small Changes

- Recall that if $f = F'$, then

$$\begin{aligned} S_f(P, C) &= S_{F'}(P, C) \\ &= \sum_{k=1}^{N} F'(c_k)(x_k - x_{k-1}) \\ &\simeq \sum_{k=1}^{N} F(x_k) - F(x_{k-1}) \\ &= F(b) - F(a) \end{aligned}$$

- In fact, by the Mean Value Theorem, we can choose each c_k such that

$$F'(c_k)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1})$$

- For this subordinate sequence,

$$S_f(P, C) = F(b) - F(a)$$

Riemann Sum as Signed Area

- The k-th term in $S_f(P, C)$,

$$f(c_k)(x_k - x_{k-1})$$

  is the area of a rectangle with width $x_k - x_{k-1}$ and height $f(c_k)$, where the height can be negative
- The sum is therefore the signed area of the region enclosed by the rectanges
- If f is continuous and rectangles are all thin, then

$$\text{Area between graph of } f \text{ and } x\text{-axis} \simeq S_f(P, C)$$

Upper and Lower Sums

- Assume that $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition of $[a, b]$
  - It follows that f is bounded on each subinterval $[x_{k-1}, x_k]$
- On each subinterval $[x_{k-1}, x_k]$, let

$$m_k = \inf\{f(x) \ : \ x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup\{f(x) \ : \ x \in [x_{k-1}, x_k]\}$$

- The **lower Riemann sum** for the partition P is

$$S_f^{\min}(P) = \sum_{k=1}^{N} m_k(x_k - x_{k-1})$$

  and the **upper Riemann sum** is

$$S_f^{\max}(P) = \sum_{k=1}^{N} M_k(x_k - x_{k-1})$$

Upper and Lower Integrals

- If $f : [a, b] \to \mathbb{R}$ is bounded, then there exists $m < M$ such that
$$\forall x \in [a, b],\ \ m \le f(x) \le M$$
- It follows that any Riemann sum satisfies
$$m(b-a) \le S_f^{\min}(P) \le S_f(P, C) \le S_f^{\max}(P) \le M(b-a)$$
- The **lower integral** of f is defined to be
$$\underline{\int_a^b} f(x)\,dx = \sup\{S_f^{\min}(P) :\ \text{all partitions } P \text{ of } [a, b]\}$$
- The **upper integral** of f is defined to be
$$\overline{\int_a^b} f(x)\,dx = \inf\{S_f^{\max}(P) :\ \text{all partitions } P \text{ of } [a, b]\}$$

Inequalities between Riemann sums

- For each $1 \leq k \leq N$, $m_k \leq f(c_k) \leq M_k$ and therefore

$$\begin{aligned} S_f^{\min}(P) &= \sum_{k=1}^{N} m_k(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{N} f(c_k)(x_k - x_{k-1}) = S_f(P, C) \\ &\leq \sum_{k=1}^{N} M_k(x_k - x_{k-1}) = S_f^{\max}(P) \end{aligned}$$

Refinement of a Partition

- A partition $Q = (y_1, \ldots, y_M)$ is a **refinement** of a partition $P = (x_1, \ldots, x_N)$ if

$$\{x_1, \ldots, x_N\} \subset \{y_1, \ldots, y_M\}$$

- If $x_{k-1} < y_j < x_k$, then

$$\inf\{f(x) \ : \ x_{k-1} \le x \le x_k\}$$
$$\le \min(\inf\{f(x) \ : \ x_{k-1} \le x \le y_j\}, \inf\{f(x) \ : \ y_j \le x \le x_k\})$$

- It follows that

$$S_f^{\min}(P) \le S_f^{\min}(Q)$$

- Similarly,

$$S_f^{\max}(P) \ge S_f^{\max}(Q)$$

Union of Two Partitions

- Given partitions $P = (x_0, \ldots, x_N)$ and $Q = (y_0, \ldots, y_M)$, we define a new partition $R = (z_0, \ldots, z_L)$ such that

$$\{z_0, \ldots, z_N\} = \{x_0, \ldots, x_N\} \cup \{y_0, \ldots, y_M\}$$

- The partition R is a refinement of both P and Q
- This partition is denoted $P \cup Q$

Lower Integral $\leq$ Upper Integral

- For any two partitions P and Q, since $P \cup Q$ is a refinement of both,

$$S_f^{\min}(P) \leq S_f^{\min}(P \cup Q) \leq S_f^{\max}(P \cup Q) \leq S_f^{\max}(Q)$$

- This implies that

$$\begin{aligned} \underline{\int_a^b} f(x)\,dx &= \sup\{S_f^{\min}(P)\colon \text{ all partitions } P\} \\ &\leq \sup\{S_f^{\min}(Q)\colon \text{ all partitions } Q\} \\ &\leq \overline{\int_a^b} f(x)\,dx \end{aligned}$$

Riemann Integral

- A function $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

- In that case, the Riemann integral of f over $[a, b]$ is

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

Properties of Riemann Integral

- Constant factor rule: If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $c \in \mathbb{R}$, then cf is Riemann integrable and

$$\int_a^b (cf)(x)\, dx = c \int_a^b f(x)\, dx$$

- Sum rule: If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are Riemann integrable, then $f + g$ is Riemann integrable and

$$\int_a^b (f + g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

- If $\forall x \in [a, b],\ m \leq f(x) \leq M$, then

$$m(b - a) \leq \int_{x=a}^{x=b} f(x)\, dx \leq M(b - a)$$

- If $\forall x \in [a, b],\ |f(x)| \leq M$, then

$$0 \leq \left| \int_{x=a}^{x=b} f(x)\, dx \right| \leq \int_{x=a}^{x=b} |f(x)|\, dx \leq M(b - a)$$