MATH-UA 325 Analysis I
Fall 2023

Fundamental Theorems of Calculus

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Updated December 11, 2023

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Integral of Derivative (Part 1)

- Let $f : [a, b] \to \mathbb{R}$ be a continuous function that is differentiable on (a, b)
- Given any partition $P = (x_0, \ldots, x_n)$ of $[a, b]$, it follows by the Mean Value Theorem that there exists $C = (c_1, \ldots, c_N)$ such that

$$f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1}),\ \forall 1 \leq k \leq N$$

- This defines a Riemann sum for f' where

$$\begin{aligned} S_{f'}(P, C) &= \sum_{k=1}^{N} f'(c_k))(x_k - x_{k-1}) \\ &= \sum_{k=1}^{N} f(x_k) - f(x_{k-1}) \\ &= f(b) - f(a) \end{aligned}$$

Integral of Derivative (Part 2)

- This shows that for any partition P, there exists $C = (c_1, \ldots, c_N)$ such that

$$S_{f'}(P, C) = f(b) - f(a)$$

- On the other hand,

$$S_{f'}^{\min}(P) \leq S_{f'}(P, C) \leq S_{f'}^{\max}(P)$$

- It follows that for any partition P,

$$S_{f'}^{\min}(P) \leq f(b) - f(a) \leq S_{f'}^{\max}(P)$$

- This implies that

$$\underline{\int_a^b} f'(x)\,dx \leq f(b) - f(a) \leq \overline{\int_a^b} f(x)\,dx$$

Fundamental Theorem of Calculus

If $I \subset \mathbb{R}$ is open, $f : I \to \mathbb{R}$ is differentiable, and f' is Riemann integrable, then for any compact interval $[a, b] \subset I$,

$$\int_{x=a}^{x=b} f'(x)\,dx = f(b) - f(a)$$

Fundamental Theorem of Calculus

- Let $f : (a, b) \to \mathbb{R}$ be Riemann integrable
- Given $x_0 \in (a, b)$, let $F : (a, b) \to \mathbb{R}$ be the function given by

$$F(x) = \int_{t=x_0}^{t=x} f(t)\, dt$$

- Then F is continuous
- If f is continuous at $x \in (a, b)$, then
    - F is differentiable at x and $F'(x) = f(x)$
    - F is the unique antiderivative of f such that $F(x_0) = 0$

Proof (Part 1)

- Since f is Riemann integrable, it is bounded
    - There exists $M > 0$ such that

$$\forall x \in (a, b),\ |f(x)| \leq M$$

- For any $x_0, x \in (a, b)$,

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{x_0}^{x} f(t)\, dt \right| \\ &\leq \left| \int_{x_0}^{x} |f(t)|\, dt \right| \\ &\leq M|x - x_0| \end{aligned}$$

- This implies that F is continuous at each $x_0 \in (a, b)$

Proof (Part 2)

- First, observe that for any $x, y \in (a, b)$,

$$\frac{F(y) - F(x)}{y - x} = \frac{1}{y - x} \int_x^y f(t)\,dt$$

- It follows that

$$\frac{F(y) - F(x)}{y - x} - f(x) = \frac{1}{y - x} \int_x^y f(t) - f(x)\,dt$$

- On the other hand, if f is continuous at x, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$$

Proof (Part 3)

- Therefore, given $x \in (a, b)$, there exists $\delta > 0$ such that if $0 < |y - x| < \delta$,

$$\begin{aligned}\left|\frac{F(y) - F(x)}{y - x} - f(x)\right| &= \left|\frac{1}{y - x}\int_x^y f(t) - f(x)\,dt\right| \\ &\le \frac{1}{|y - x|}\int_x^y |f(t) - f(y)|\,dt \\ &\le \epsilon\end{aligned}$$

- In other words,

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x} - f(x) = 0$$

- I.e.,

$$F'(x) = f(x)$$

Uniform Continuity

- Recall that a function $f : I \to \mathbb{R}$ is **continuous** if for any $\epsilon > 0$ and $x_0 \in I$, there exists $\delta_{x_0} > 0$ such that for any $x_1 \in I$,

$$|x_0 - x_1| < \delta \implies |f(x_0) - f(x_0)| < \epsilon$$

- A function $f : I \to \mathbb{R}$ is **uniformly continuous** if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_0, x_1 \in I$,

$$|x_0 - x_1| < \delta \implies |f(x_0) - f(x_1)| < \epsilon$$

Continuity on Compact Set Implies Uniformly Continuity

- Suppose $f : [a, b] \to \mathbb{R}$ is continuous but not uniformly continuous
- Not uniformly continuous means that there exists $\epsilon > 0$ such that for any $\delta > 0$, there exists $x, y \in I$ such that

$$0 < |x - y| < \delta \text{ and } |f(x) - f(y)| > \epsilon$$

- It follows that for each $n \in \mathbb{N}$, there exists x_n, y_n such that

$$a \leq x_n < y_n \leq b \text{ and } |f(x_n) - f(y_n)| > \epsilon.$$

Continuous Implies Riemann Integrable

- Let $f : [a, b] \to \mathbb{R}$ be continuous and therefore uniformly continuous
- Given a partition $P = (x_0, \ldots, x_N)$ of $[a, b]$, let

$$w(P) = \max(x_1 - x_0, x_2 - x_1, \ldots, x_N - x_{N-1})$$

- For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

- For each $1 \leq k \leq N$, $m_k = f(c_{\text{min}})$ and $M_k = f(c_{\text{max}})$ for some $c_{\text{min}}, c_{\text{max}} \in [x_{k-1}, x_k]$
- Since

$$|c_{\text{min}} - c_{\text{max}}| < |x_k - x_{k-1}| < \delta$$

it follows that

$$M_k - m_k = |f(c_{\text{max}}) - f(c_{\text{min}})| < \epsilon$$