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Exponential Functions

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Affine Functions on $\mathbb{R}$

- A function $f : \mathbb{R} \to \mathbb{R}$ is **affine** (but often called linear) if the change in output, depends only on the change in input and not on the input itself
- I.e., for each change in input, $\Delta \in \mathbb{R}$, there exists $c(\Delta) \in \mathbb{R}$ such that

$$f(x + \Delta) - f(x) = c(\Delta), \ \forall x \in \mathbb{R}$$

- If f is assumed to be differentiable, then

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x + \Delta) - f(x)}{\Delta} = \lim_{\Delta \to 0} \frac{c(\Delta)}{\Delta} = c'(0)$$

- This implies that for any $x, \Delta \in \mathbb{R}$,

$$f(x) = mx + b \text{ and } f(x + \Delta) - f(x) = m\Delta,$$

where $m = c'(0)$ and $b = f(0)$

Exponential Functions

- A function $f : \mathbb{R} \to \mathbb{R}$ is called **exponential**, if the **percentage or relative change** in output depends only on the change in input and not on the input itself
- I.e., for each change in input, $\Delta \in \mathbb{R}$, there exists $c(\Delta) \in \mathbb{R}$ such that

$$\frac{f(x+\Delta) - f(x)}{f(x)} = c(\Delta), \ \forall x \in \mathbb{R} \tag{1}$$

- Exponential requires $f(x) \neq 0$ for all $x \in \mathbb{R}$
- If f is differentiable, then

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta} = \lim_{\Delta \to 0} \frac{c(\Delta)}{\Delta} f(x) = c'(0) f(x)$$

- **Conclusion:** If a function f is exponential, then there is constant κ such that

$$f' = \kappa f$$

Existence and Uniqueness of Exponential Functions

- **Theorem:** Given $E_0, \kappa \in \mathbb{R}$, there exists a unique differentiable function $E : \mathbb{R} \to \mathbb{R}$ such that

$$E' = \kappa E \text{ and } E(0) = E_0 \tag{2}$$

- For each $\kappa \in \mathbb{R}$, let $e_\kappa : \mathbb{R} \to \mathbb{R}$ be the unique differentiable function such that

$$e_\kappa' = \kappa e_\kappa \text{ and } e_\kappa(0) = 1$$

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- The standard exponential function is defined to be $e_1 : \mathbb{R} \to \mathbb{R}$, which satisfies

$$e_1' = e_1 \text{ and } e_1(0) = 1$$

Existence of Exponential Functions

- It suffices to prove the existence of the standard exponential function e_1
- For any $\kappa, E_0 \in \mathbb{R}$, if

$$E(t) = E_0 e_1(\kappa t),\ \forall t \in \mathbb{R},$$

  then $E' = \kappa E$ and $E(0) = E_0$
- Solve for e_1 as a power series

Existence of Standard Exponential Function

- If
$$e_1(x) = \sum_{k=0}^{\infty} a_k x^k \text{ and } e_1'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1},$$
then $a_0 = e_1(0) = 1$ and
$$\begin{aligned} e_1'(x) - e_1(x) &= \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} (k+1) a_k x^k - \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{k=0}^{\infty} ((k+1) a_{k+1} - \kappa a_k) x^k \end{aligned}$$

- Therefore, if $e_1' = e_1$, then for any $k \geq 0$,
$$(k+1) a_{k+1} = a_k \text{ and, by induction, } a_k = \frac{1}{k!}$$

Power Series for e_1 Converges Absolutely

- The power series for e_1 is

$$\sum_{k=0} \frac{x^k}{k!}$$

- If $s_k = \dfrac{x^k}{k!}$, then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{s_{k+1}}{s_k}\right| &= \lim_{k\to\infty}\frac{|x|^{k+1}}{(k+1)!}\frac{k!}{|x|^k} \\ &= \lim_{k\to\infty}\frac{|x|}{k+1} \\ &= 0,\end{aligned}$$

- By the ratio test, this implies the power series converges absolutely for all $x \in \mathbb{R}$

Definition of e_1

- We now **define** the exponential function e_1 to be

$$e_1(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}, \ \forall x \in \mathbb{R}$$

- We now need to show that e_1, as defined above, satisfies the equation $e_1' = e_1$

$e_1' = e_1$

- Observe that, since the power series for e_1 converges absolutely,

$$\frac{e_1(y) - e_1(x)}{y - x} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{y^k - x^k}{y - x}$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!}(y^{k-1} + y^{k-2}x + \cdots + x^{k-1})$$

- Therefore,

$$e_1'(x) = \lim_{y \to x} \frac{e_1(y) - e_1(x)}{y - x}$$
$$= \lim_{y \to x} \sum_{k=1}^{\infty} \frac{1}{k!}(y^{k-1} + y^{k-2}x + \cdots + x^{k-1})$$
$$= \sum_{k=1} \frac{kx^{k-1}}{k!} = \sum_{k=0} \frac{x^k}{k!} = e_1(x)$$

Exponential Function is Always Positive (Part 1)

- Given $E_0 > 0$ and $\kappa > 0$, let $E : \mathbb{R} \to \mathbb{R}$

$$E' = \kappa E \text{ and } E(0) = E_0$$

- If there exists $t > 0$ such that $E(t) = 0$, then let

$$T = \inf\{t \ : \ E(t) = 0\}$$

- It follows that for any $t \in [0, T)$, $E'(t) = \kappa E(t) > 0$
- Therefore, E is strictly increasing on $[0, T)$
- Since E is continuous on $\mathbb{R}$, it follows that $E(T) > E(0) > 0$
- It follows that no such T exists

Exponential Function is Always Positive (Part 2)

- If
$$f(t) = \frac{1}{E(-t)},$$
then
$$f'(t) = -\frac{1}{(E(-t))^2}(-E'(-t)) = \frac{\kappa E(-t)}{(E(-t))^2} = \kappa f(t)$$
and
$$f(0) = \frac{1}{E_0} > 0$$
- It follows that $f(t) > 0$ for all $t \geq 0$ and therefore $E(t) > 0$ for all $t \leq 0$

Uniqueness of Exponential Functions

- Suppose E_1, E_2 both satisfy $E_1(0) = E_2(0) = E_0$,

$$E_1' = \kappa E_1, \text{ and } E_2' = \kappa E_2$$

- Then

$$\left(\frac{E_1}{E_2}\right)' = \frac{E_2 E_1' - E_1 E_2'}{E_2^2} = \frac{E_1}{E_2}\left(\frac{E_1'}{E_1} - \frac{E_2'}{E_2}\right) = \frac{E_1}{E_2}(\kappa - \kappa) = 0$$

- Since $E_1(0) = E_2(0)$, it follows that

$$\frac{E_1}{E_2} = 1, \text{ i.e., } E_1 = E_2$$

Translation Invariance of Exponential Functions

- For each exponential function E and $s \in \mathbb{R}$, consider the function $E_s : \mathbb{R} \to \mathbb{R}$ where for each $t \in \mathbb{R}$,

$$E_s(t) = E(s + t)$$

- E_s is itself an exponential function, because

$$E_s' = \kappa E_s,\ E_s(0) = E(s)$$

- On the other hand, the function $f_s : \mathbb{R} \to \mathbb{R}$ given by

$$f_s(t) = \frac{E(s)E(t)}{E(0)},\ \forall t \in \mathbb{R},$$

  also satisfies

$$f_s' = \kappa f_s \text{ and } f_s(0) = E(s)$$

- By the uniqueness of exponential functions, $E_s = f_s$
- It follows that for any exponential function E and $s, t \in \mathbb{R}$,

$$E(s + t)E(0) = E(s)E(t)$$

Relative Change in Output of Exponential Functions

- We now want to show that any exponential function E satisfies the original property that the relative change in output depends only on the change in input and not on the input
- For any exponential function E and $x, \Delta \in \mathbb{R}$,

$$\frac{E(x+\Delta) - E(x)}{E(x)} = \frac{E(x)E(\Delta) - E(x)}{E(x)} = E(\Delta) - 1.$$

- Therefore, E satisfies (1) if

$$c(\Delta) = E(\Delta) - 1.$$

- It follows that if E is differentiable, then E satisfies (1) if and only if it satisfies (2).

Definition of e

- Define Euler's number to be the constant

$$e = e_1(1)$$

$e_1(k)$ for $k \in \mathbb{Z}$

- For any integer k and $x \in \mathbb{R}$, it follows by translation invariance,

$$e_1(k) = e_1((k-1)+1) = e_1(k-1)e_1(1) = (e)e_1(k-1)$$

- Since $e_1(0) = 1$, it follows by induction that for any nonnegative integer k,

$$\forall x \in \mathbb{R} \text{ and } k \in \mathbb{Z},\ \ e_1(k) = e^k$$

- Since

$$1 = e_1(0) = e_1(k + (-k)) = e_1(k)e_1(-k) = e^k e_1(-k),$$

  it follows that for any nonnegative integer k,

$$e_1(-k) = \frac{1}{e^k} = e^{-k}$$

$e_1(r) = e^r$ **for** $r \in \mathbb{Q}$

- Any $r \in \mathbb{Q}$ can be written as
$$r = \frac{p}{q} \text{ where } p, q \in \mathbb{Z}, \text{ where } q > 0$$
- it follows by translation invariance
$$\begin{aligned} e^p &= e_1(p) \\ &= \left(e_1\left(\frac{p}{q} + \cdots + \frac{p}{q}\right)\right) \\ &= e_1\left(\frac{p}{q}\right) \cdots e_1\left(\frac{p}{q}\right) \\ &= \left(e_1\left(\frac{p}{q}\right)\right)^q \end{aligned}$$
- Therefore,
$$e_1\left(\frac{p}{q}\right) = e^{\frac{p}{q}}, \text{ i.e.,} e_1(r) = e^r \text{ for any } r \in \mathbb{Q}$$

Definition of e^x

- For any $r \in \mathbb{Q}$, $e_1(r) = e^r$
- For any $x, y \in \mathbb{R}$, $e_1(x+y) = e_1(x)e_1(y)$
- $e_1(0) = 1$
- Therefore, it is natural to write the function e_1 as

$$e^x = e_1(x),\ \forall x \in \mathbb{R}$$

- This is the **definition** of e^x for any real x
- We still do not have a definition of the function a^x, where $a \neq e$

Properties of e^x

- Since $e^x = e_1(x)$,

$$\begin{aligned} e^x &> 0,\ \forall x \in \mathbb{R} \\ e^0 &= 1 \\ e^{x+y} &= e^x e^y,\ \forall x, y \in \mathbb{R} \\ \frac{d}{dx} e^x &= e^x,\ \forall x \in \mathbb{R} \end{aligned}$$

Exponential Function is Invertible

- Since $e_1 : \mathbb{R} \to (0, \infty)$ is strictly increasing, it is injective
- Since $e > 1$,

$$\lim_{k \to \infty} e^{-k} = 0,$$

  and the sequence $(e^k : \ k \in \mathbb{Z}_+)$ is unbounded
- This implies that $e_1 : \mathbb{R} \to (0, \infty)$ is surjective

Natural Logarithm Function

- The **natural logarithm function** is defined to be the inverse function of e_1 and denoted

$$\begin{aligned} \ln : (0,\infty) &\to \mathbb{R} \\ x &\mapsto \ln(x) \end{aligned}$$

- Basic properties
  - Since $e^0 = 1$,

$$\ln(1) = 0$$

  - Since $e^{x+y} = e^x e^u$,

$$\ln(ab) = \ln a + \ln b, \ \forall a, b \in (0,\infty)$$