MATH-UA 325 Analysis I
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Complex Exponential Functions

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Quick Introduction of Complex Numbers (Part 1)

- Let i be a new type of number that satisfies the property that

$$i^2 = -1$$

- A complex number is something of the form $a + ib$, where $a, b \in \mathbb{R}$
- The set of all complex numbers is denoted $\mathbb{C}$
- **Complex addition:**

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

- **Complex multiplication:** Using the commutative and distributive properties,

$$(a+ib)(c+id) = ac+a(id)+(ib)c+(ib)(id) = (ac-bd)+i(ad+bd)$$

Quick Introduction of Complex Numbers (Part 2)

- Define **conjugate** of a complex number $z = x + iy$ to be

$$\overline{z} = a - ib$$

- Observe that

$$z\overline{z} = \overline{z}z = (x + iy)(x - iy) = x^2 + y^2$$

- Define the absolute value or **norm** of $z = x + iy \in \mathbb{C}$ to be

$$\begin{aligned} |z| &= \sqrt{z\overline{z}} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

Quick Introduction of Complex Numbers (Part 3)

- The **reciprocal** of $z = a + ib \neq 0$ is denoted z^{-1} and satisfies

$$1 = zz^{-1}$$

This implies

$$\begin{aligned} \overline{z} &= \overline{z}zz^{-1} \\ &= |z|^2 z^{-1}, \end{aligned}$$

which implies

$$\begin{aligned} z^{-1} &= \frac{\overline{z}}{|z|^2} \\ &= \frac{a - ib}{a^2 + b^2} \end{aligned}$$

- If $z, w \in \mathbb{C}$ and $w \neq 0$, define their **quotient** to be

$$\frac{z}{w} = zw^{-1} = \frac{z\overline{w}}{|w|^2}$$

Complex Plane

- There is a bijection

$$\mathbb{R}^2 \to \mathbb{C}$$
$$(x, y) \mapsto x + iy$$

- Addition of complex numbers is equivalent to vector addition on $\mathbb{R}^2$
- Multiplication of a complex number by a real number is equivalent to scalar multiplication of a vector in $\mathbb{R}^2$
- Complex multiplication is not a property of the vector space $\mathbb{R}^2$

Complex-Valued Functions

- A **complex-valued function** is a function of the form

$$f : \mathbb{R} \to \mathbb{C}$$

- Any such function can be written $f = a + ib$, where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$
- Examples
  - Any real-valued function $f : \mathbb{R} \to \mathbb{R}$
  - Given any $c \in \mathbb{C}$, the constant function $f(z) = c$ for all $z \in \mathbb{C}$
  - $f(z) = a_0 + a_1 z + \cdots a_d z^d$
- A complex-valued function $f = a + ib$ is differentiable if a, b are differentiable
- If $f = a + ib$ is differentiable, then $f' = a' + ib'$

Exponential of an Imaginary Number

- $z = x + iy$ is **imaginary** if $x = 0$ and $z = iy$
- There is a unique complex-valued function $e_i : \mathbb{R} \to \mathbb{C}$ such that
$$\forall t \in \mathbb{R},\ e_i'(t) = ie_i(t) \text{ and } e_i(0) = 1$$
- Using the same notation as the real exponential function, we will write
$$e^{it} = e_i(t)$$
- e_i satisfies the same properties as the real exponential e_1:
$$\begin{aligned} e^{it} &\neq 0,\ \forall t \in \mathbb{R} \\ e^{i0} &= 1 \\ e^{i(s+t)} &= e^{is}e^{it},\ \forall s, t \in \mathbb{R} \\ e^{-it} &= \frac{1}{e^{it}} \end{aligned}$$

Geometry of e^{it} (Part 1)

- We can write $e^{it} = x(t) + iy(t)$
- Its derivative is, on one hand,

$$\frac{d}{dt}e^{it} = x'(t) + iy'(t)$$

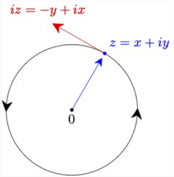
- On the other hand, by the definition of e^{it},

$$\frac{d}{dt}e^{it} = ie^{it} = i(x(t) + iy(t)) = -y(t) + ix(y)$$

- Therefore,

$$\frac{d}{dt}(x^2 + y^2) = 2xx' + 2yy' = 2(-xy) + 2(yx) = 0$$

Geometry of e^{it} (Part 2)



- Since $x(0) + iy(0) = e^{i0} = 1$ and

$$\frac{d}{dt}(x^2 + y^2) = 0,$$

it follows that

$$(x(t))^2 + (y(t))^2 = 1, \; \forall t \in \mathbb{R}$$

- The velocity vector is always perpendicular to the radial vector, because

$$(-y, x) \cdot (x, y) = -yx + xy = 0$$

Definition of π

- Translation invariance can be used to prove that the function $t \to e^{it}$ goes around the unit circle at least once
- In other words, there exists $T > 0$ such that

$$e^{iT} = 1$$

- The constant $\pi \in \mathbb{R}$ is defined to be the smallest positive constant such that

$$e^{i2\pi} = 1$$

- It follows that for any $t \in \mathbb{R}$,

$$e^{i(t+2\pi)} = e^{it}$$

Definition of Sine and Cosine Functions

- There are functions x and y such that

$$e^{it} = x(t) + iy(t),\ \forall t \in \mathbb{R}$$

- The sine and cosine functions are defined to be

$$\cos(t) = x(t) \text{ and } \sin(t) = y(t)$$

- I.e., they are the functions satisfying

$$e^{it} = \cos(t) + i\sin(t)$$

- All of the properties and identities of the sine and cosine functions can be derived using the properties of e^{it} proved above and translation invariance

Definition of Radians

- For each point (x, y) on the unit circle, there exists a unique $\theta \in [0, 2\pi)$ such that

$$e^{i\theta} = x + iy$$

- The angle between the positive x-axis and the ray passing through (x, y) is defined to be θ radians
- The angle from $e^{i\theta_1} = x_1 + iy_1$ to $e^{i\theta_2} = x_2 + iy_2$ is defined to be $\theta_2 - \theta_1$ radians

Euler's Identity

- Observe that
$$(e^{i\pi})^2 = e^{i2\pi} = 1$$
- Therefore, $e^{i\pi} = 1$ or -1
- The definition of π says that $T = 2\pi$ is the smallest $T > 0$ such that
$$e^{iT} = 1$$
- Therefore,
$$e^{i\pi} = -1,$$
i.e.,
$$e^{i\pi} + 1 = 0$$