

Computer Exercises HW 7

Objective: The QR factorization based on the pivoted Gram-Schmidt orthogonalization. Use format long e to present numerical results.

1. Given $A \in \mathbb{R}^{m \times n}$ as input, construct a Matlab function `[Q,R,permu,rank]=pivotgs(A,eps0)` to implement the Modified Gram-Schmidt with column pivoting, such that `permu` is the permutation vector for the pivoting process, and that $A(:,\text{permu})=Q \cdot R + 0(\text{eps0})$. Note that the size of `Q` must be `m-by-rank`, `R` must be `rank-by-rank`, so that they are the reduced form of QR factorization. The role of `eps0` in pivoting is as follows.

a. Initially, `permu=1:n`, a row vector.

b. For each $k = 1 : n$, we are at the k -th step of the Modified G-S, and we process the k -th column of the current `A` which is not `A(:,k)` but is `A(:,permu(k))`. In other words, we never actually move or exchange the columns of `A` in pivoting; rather, we record the permutation process in the array `permu`, and use it to index the columns of matrix `A`.

b1. Next, pick a column out of the remaining k -th to n -th columns `A(:,permu(k:n))` that has the largest norm among them. Let's say that this chosen column is `A(:,permu(l))` for some l between k and n . If $\text{norm}(A(:,\text{permu}(l)))$ is smaller than $\text{eps0} * \text{norm}(A, 'fro')$, stop and output `rank=k-1`, `Q`, and `R`. In other words, we treat the remaining columns as zeros because they are below our threshold `eps0`.

b2. Swap this column with the k -th column: Don't actually swap them, instead swap the array `permu`.

b3. Don't forget to swap the same two columns of matrix `R`: This time actually swap them.

c. Obtain `Q(:,k)` – the k -th column of matrix `Q`, and proceed as before when we constructed the Modified Gram-Schmidt without column pivoting.

Remark. In step b1, you'll find `A(:,permu(k:n)).^2`, and the functions `sum`, `max` useful for vectorization and simplicity. In particular, `max` as a function applied to a vector not only returns the maximum value of the vector entries but also the index at which the maximum occurred.

(i) Provide Matlab script for the function

(ii) Run it on the the matrix `A=hilb1(m,n)` with $m=15$, $n=10$, `eps0=1.0e-6`. Calculate and show the two backward errors

$$\|Q^T Q - I\|_2 \tag{1}$$

$$\frac{\|Q R - A\|_2}{\|A\|_2} \tag{2}$$

(iii) Show the diagonal entries of `R`; do they appear in a certain order.

2. Use the QR factorization `[Q,R,permu,rank]=pivotgs(A,eps0)` which you constructed and tested in Problem 1 to build an orthogonal basis for the harmonic wave functions

$$\mathcal{W} = \text{span}_{\alpha} \{ \sin(\alpha \cdot t) \mid t \in [-\pi, \pi], \alpha \in [1, 20] \} \tag{3}$$

where \mathcal{W} denotes this function class; The letter \mathcal{W} could stand for waves, or for whales because the frequencies α are so low, below 20 Hertz, at which the species is certainly capable of wailing. Each function $f \in \mathcal{W}$ is regarded as a function of t , thus the span is done with different values of α . Such a function could be thought of as an acoustic signal. In order to form a matrix A to represent the space \mathcal{W} , we sample its functions with finite number of points. More precisely, let $\alpha = [1:0.1:20]$, namely, sample α with 191 equispaced points. Let $\mathbf{t} = [-\pi:2*\pi/600:\pi]$, namely, sample \mathbf{t} it with 601 equispaced points.

- (i) What is your choice of the matrix A . Hint: it is a 601-by-191 matrix
- (ii) Perform the QR factorization `[Q,R,permu,rank]=pivotgs(A,eps0)` with `eps0=1.0e-5`. Show the numerical rank `rank` for the prescribed precision `eps0=1.0e-5`. Show the diagonal elements of R ; there are `rank` of them.
- (iii) Plot the first four basis functions `Q(:,1:4)` using subplot method: four subplots on a single paper; they are functions of `t = [-pi:2*pi/600:pi]`.
- (iv) Plot the 17-th to 20-th basis functions: 4 subplots on a single paper.
- (v) Plot the four columns `Q(:,rank-3:rank)`: 4 subplots on a single paper.