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Homework 9

Objective: Newton's iteration to solve nonlinear equations. Numerical solution of nonlinear optimization problems.

1. In order to find the square root of a positive number $a > 0$, solve the nonlinear equation $f(x) =: x^2 - A = 0$ with Newton's iteration.

(i) Write down Newton's iteration for this function.

(ii) Implement it with a Matlab function root=mysqrt(A,eps0). Do not provide your code, but do have a suitable stop criterion built in the function which guarantees double precision eps0=2.0e-16. Provide your stop criterion.

(iii) Choose your own initial guess to find roots for the three cases A=3, A=1.0e40 * pi, and A=exp(-200). Note that you must first compute A according to these three formulae and then send each of them to the function mysqrt(A,eps0). For each root finding process, show the relative errors for every Newton iteration (as compared to rootexact=sqrt(A), which by itself is not exact, it is accurate to 16 digits).

2. The Schult's method for inverting an n-by-n matrix A is described in Problem 5.28, page 254. Do part (a) of this problem; namely, prove

$$R_{k+1} = R_k^2, \quad E_{k+1} = E_k\, A\, E_k \tag{1}$$

3. Denote by $J_0(z)$ the Bessel J function of order 0; in Matlab this function is computed by `nu=0; bj=besselj(nu,z)`. Find three real numbers a, b, c to minimize the quantity

$$F(a,b,c) =: \int_\alpha^\beta \left[J_0(z) - a\,\cos(b\,z + c)\right]^2 dz, \quad \alpha = 0,\ \beta = 2\pi \tag{2}$$

using Newton's iteration. In order to numerically solve this problem (which involve an integral which we don't know how to do analytically), we must discretize the integral: To replace it with a finite sum. This can be done before writing down the Newton's iteration formula or after it. We'll discretize before it; namely we use $n = 40$ equispaced points

$$\{\, z_i = \alpha + i\,h \mid h = (\beta - \alpha)/n,\ i = 1, 2, \ldots, n \,\} \tag{3}$$

to rewrite the original problem (2) as

$$\min\ f(a,b,c) =: h \sum_{i=1}^{n} \left[J_0(z_i) - a\,\cos(b\,z_i + c)\right]^2, \tag{4}$$

a. Plot the function $J_0(z)$ in $[\alpha, \beta]$.

b. Find $a,\ b,\ c$ to solve the optimization problem (4) by finding the roots of the gradient of f. Namely, solve the three equations $\nabla f(a,b,c) = 0$ by Newton's iteration (denote a root by $(\bar a, \bar b, \bar c)$, a point in 3-D)

c. Provide the minimum value f and the gradient ∇f at $(\bar a, \bar b, \bar c)$. For this $(\bar a, \bar b, \bar c)$, plot the error function

$$e(z) = J_0(z) - u(z), \quad \text{where} \quad u(z) = \bar a\,\cos(\bar b\,z + \bar c). \tag{5}$$

d. Find as many solutions (local minima) as you can, and for each of them do part (c). Which one gives rise to the global minimum of our problem?