New York University
Courant Institute of Mathematical Sciences

Problem Set 3

Problem 1

Exercise 3.5, page 51.

Problem 2

The purpose of this problem is to demonstrate the superiority of high order methods for ODEs in terms of their efficiency and accuracy. We take the simplest equation

$$\begin{cases} y'(t) = i \cdot k \cdot y(t), & 0 \leq t \leq 2\pi, \\ y(0) = 1, \end{cases} \tag{1}$$

and the simple method – the ν-stage ERK method of order ν, as is considered in Problem 1. What you observe in the experiment you are asked to do is applicable for more general ODEs and more sophisticate schemes.
(a) For $k = 100$, and for each $\nu = 1, 2, 3, 4, 8, 12, 16, 20$, solve (1) with the ν-stage ERK method to 5-digit precision, namely, the relative error of solution at $t = 2\pi$ is no greater than 10^{-5}.
(b) Find (numerically) the smallest number n of equispaced points over $[0, 2\pi]$ needed to achieve this precision for each ν. Divide the number n by k to get the (minimum) number of points per wavelength needed for the precision. Tabulate these numbers against ν.
(c) Repeat (a) and (b) for $k = 1000$.

Problem 3

It is well-known that the IRK methods based on Gaussian points are all A-stable (see, e.g., Section 4.3, page 59–63). Write down explicitly the growth factor of the fourth order (two-stage) IRK method given on page 47 (which you tested in the previous assignment). Show that
(a) The method is indeed A-stable.
(b) Is the method L-stable, and why?

Problem 4

Given a simple argument to explain that there is no contradiction between the two facts.
(i) $z = (1 + ikh)$ is never inside the stability domain $|1 + z| \leq 1$ of the (forward) Euler's method (which usually means unstable solution, or explosive growth of the numerical solution).
(ii) Euler's method converges (Theorem 1.1) for the equation $y'(t) = iky(t)$.