New York University
Courant Institute of Mathematical Sciences

G22.2112-001/G63.2043 — Scientific Computing — *Handout #4* — *Feb. 15, 2001*

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Assignment 4

1. Write a Matlab script of no more than 10 lines to calculate the Newton-Cotes quadrature weights w_j, $j = 1, 2, \ldots, n, n+1$; go to the end of this assignment for more details about w_j. For uniformity, we require that the script starts with the four lines `n=2; h=1/n; nodes=(0:h:1)'; f=eye(n+1);` where the j-th column of the matrix `f` are needed for the Lagrange interpolation in order to find the Lagrange Basis functions $L_{n,j}$; see bottom for more details.

- **a.** Provide the script.
- **b.** Print off w (to 16 digits) for $n = 1$ (trapezoidal rule), $n = 2$ (Simpson's rule), $n = 7$ (call it border-line rule), $n = 8$ (call it misery's rule) and n=20 (call it the Devil's rule)
- **c.** For each rule, print out the Lebesgue constant = norm(h*w,1)

Hint: Make use of the three Matlab functions polyval, polyint, and polyfit in the order `polyval(polyint(polyfit( ··· )))` . For those whose Matlabs don't support these functions, you need to find a Matlab which supports them; it is almost impossible to do this problem with hand.

2. Use the Matlab function fzero to write a code solving the nonlinear equation for α - the order of convergence (Do not provide the code).

- **a.** Give the nonlinear equation for α in the form $f(\alpha) = 0$ which is used in your implementation (this equation may not be unique, so just give your version)
- **b.** Check the order of the algorithm `diffgen` when it calculates the first derivative of $x^2 \cos(x)$ at $x = 1$ with $h_1 = 0.01$, $h_2 = 0.014$ $h_3 = 0.008$.

3. A Quiz.

- **a.** Suppose that a polynomial of degree 200 has 100 real and simple roots in the interval $[-10, 10]$, separated one from another by a distance of at least 4×10^{-3}. Describe a simple and reliable procedure to find all the real roots to double precision (no actual programming is required).
- **b.** Write down the analytic expression for the n Chebyshev nodes in the interval $[1, 2]$
- **c.** Find out an integer that is the minimum number of Chebyshev nodes required for interpolating the Natural Logarithm $\ln(x)$ in the interval $[2, 4]$ to precision 10^{-12}.

4. Let $\{x_j, j = 1, 2, \ldots, n\}$ be the n roots in $(-1, 1)$ of the Legendre polynomial $p_n(x)$ of degree n. The Chebyshev node $t_j = \cos[(n - j + 1/2) \cdot \pi/n]$ is known to be close to x_j. Apply Newton's iteration to find x_j to double precision with t_j as the starting value, for $j = 1, 2, \ldots, \lfloor n/2 \rfloor$.

Remark : Use the recursion $p_0(x) = 1$, $p_1(x) = x$, and $p_{k+1}(x) = [(2k+1) \cdot x \cdot p_k(x) - k \cdot p_{k-1}(x)]/(k+1)$ to evaluate $p_n(x)$; derive a similar recursion to evaluate $p'_k(x)$.

- **a.** For $n = 6$ and 11, make two plots marking the locations of the points x_j and t_j for $j = 1, 2, \ldots, \lfloor n/2 \rfloor$.
- **b.** Show the quadratic convergence of the iterations for the case $n = 11$, $j = 3$.
- **c.** Describe in plain English your stopping criterion

d. Show the number of iterations required to obtain double precision solution for the case $n = 11,\ j = 1, 2, 3, 4, 5$.

Lagrange Interpolation. The j-th Lagrange Basis function associated with the $n + 1$ distinct nodes $\{\ x_k,\ k = 0, 1, \ldots, n\ \}$ is defined by the formula

$$L_{n,j}(x) = \prod_{k \neq j} \frac{x - x_k}{x_j - x_k} \tag{1}$$

for $j = 0, 1, \ldots, n$. The interpolating polynomial $P_n(x)$ to a function $f(x)$ at the nodes $\{x_j\}$ is given by the formula

$$P_n(x) = \sum_{j=0}^{n} f(x_j) L_{n,j}(x) \tag{2}$$

In the associated quadrature with nodes $\{x_j\}$

$$\int_a^b f(x)dx \sim \int_a^b P_n(x)dx = \sum_{j=0}^{n} w_j f(x_j), \tag{3}$$

the weights are obviously given by

$$w_j = \int_a^b L_{n,j}(x)dx \tag{4}$$

In the special case of Newton-Cotes, $a = x_0,\ b = x_n$, and $\{x_j\}$ are equispaced. Moreover, the weights w_j are traditionally defined as

$$w_j = \frac{1}{h} \int_a^b L_{n,j}(x)dx \tag{5}$$

and consequently, the quadrature assumes the form

$$\int_a^b f(x)dx \sim h \sum_{j=0}^{n} w_j f(x_j) \tag{6}$$