New York University
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G22.2112-001/G63.2043 Scientific Computing
Handout #3
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Midterm Project 1.1

Objective: Solution of linear system with large condition number. Use of SVD for the solution of the least-squares problems. Optimal design of digital filters and high order quadrature formulae.

1. Introduction

A digital filter (see Figure 1 at the end) is a function $f(x)$, $x \in [a, b]$ which provides a monotone, smooth transition from the state 0 to state 1; in other words, we require that for a positive ℓ

$$f(a) = 0,\ f(b) = 1,\ f'(x) \geq 0, \quad \text{(monotone in [a, b])} \tag{1}$$

$$f \in \mathbb{C}^{\ell}[a, b], \quad \text{(smooth in [a, b])} \tag{2}$$

$$\frac{d^j f}{dx^j}(a) = 0,\ \frac{d^j f}{dx^j}(b) = 0,\ j = 1, 2, \ldots, \ell \quad \text{(smooth transition at } a \text{ and } b\text{)} \tag{3}$$

Let's from now on assume that

$$a = -\pi, \quad b = \pi. \tag{4}$$

A good filter requires a reasonably large ℓ, say, greater than 10, to ensure a smooth transition between 0 and 1. As examples, the functions

$$f_0(x) = \frac{x + \pi}{2\pi}, \quad x \in [-\pi, \pi] \tag{5}$$

$$f_1(x) = \frac{1}{2}\left[1 - \cos\left(\frac{x + \pi}{2}\right)\right] \quad x \in [-\pi, \pi] \tag{6}$$

are possible choices for a filter. But f_0 lacks the smoothness; it touches the floor 0 and the celling 1 like a stick, and Condition (3) is satisfied only for $\ell = 0$. This makes f_0 unsuitable as a filter. f_1 satisfies all Conditions (1)–(3) with $\ell = 1$, and therefore could be used as a filter (the oil industries use it for signal processing of their seismic data). But this filter delivers inferior sound quality if used in making CD's, partially because of the low ℓ value, and partially because the "physics" is not correctly built in.

This project consists of two or three parts. In this first part, we design a filter with higher ℓ value, say, 10. In the second part that is to come, you'll learn how to put in "physics" properly by doing linear algebra.

A higher ℓ value is also needed for the so-called quadrature: a summation formula to approximate integrals. As we know, the most fundamental operation in scientific computing is the inner product. Nature does it in the form of integrals. So calculating an integral to a good precision is a bread-butter issue in scientific computing.

This project is not a difficult one. It only requires a bit of calculus in the first part, and bit of linear algebra in the second. And if some of you are further interested, we may have a third part which requires a bit of symbolic computing with matlab. Symbolic computing or precessing is a powerful tool for its ability of manipulating algebraic expressions and its variable precision arithmetic. Those of you who are familiar with Maple or the like know how useful this Matlab toolbox is.

You'll use only SVD to solve a linear system whether the matrix is square and invertible or not. This is because in the second part, a non-standard least-squares problem needs to be solved which can only be handled properly with SVD.

2. DESCRIPTION OF THE PROBLEM

We will represent our filter, denoted by f_3, as a sum of two functions f_0 and p

$$f_3(x) = f_0(x) + p(x), \quad x \in [-\pi, \pi] \tag{7}$$

$$p(x) = \sum_{j=1}^{n} c_j sin(jx), \quad x \in [-\pi, \pi] \tag{8}$$

where p is a periodic function in $[-\pi, \pi]$, and the Fourier coefficients c_j are to be determine so as to satisfy Condition (3). Note that Conditions (1), (2) are already fulfilled, except the requirement $f'(x) \geq 0$ which we'll consider in the second part of the project.

It is easy to verify that Condition (3) translates to

$$p'(\pi) = -\frac{1}{2\pi}, \tag{9}$$

$$p^{(2i-1)}(\pi) = 0, \quad i = 2, 3, \ldots, m \tag{10}$$

where $m = \lfloor \ell/2 \rfloor$, since $p^{(2i)}(\pi)$ is automatically zero.

3. STATEMENT OF WORK

It is easy to see that the m equations in (9), (10) set up a system of linear equations for the n unknows $c = [c_1, c_2, \ldots, c_n]^T$; let's denote the system by $Ac = b$. Your work is to complete the following steps, solve the linear equations, and make a plot of the filter f_3.

a. Write down expressions for $A(i, j)$, $b(i)$ for the m-by-n matrix A and the RHS b, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

b. Write a matlab script for the solution of the linear system via SVD, under no assumption on m and n as which is greater than the other. Then solve for the unknown c in each of the three cases (i) $m = n = 5$ (ii) $m = n = 9$ and (iii) $m = 5$, $n = 9$.

c. Now with the Fourier coefficients c available, evaluate $f_3(x)$ and plot it with 50 equispaced x points in $[-\pi, \pi]$, for each of the three cases.

d. Brief remarks (no longer than 120 words) on the results, as what is right and what may have gone wrong.

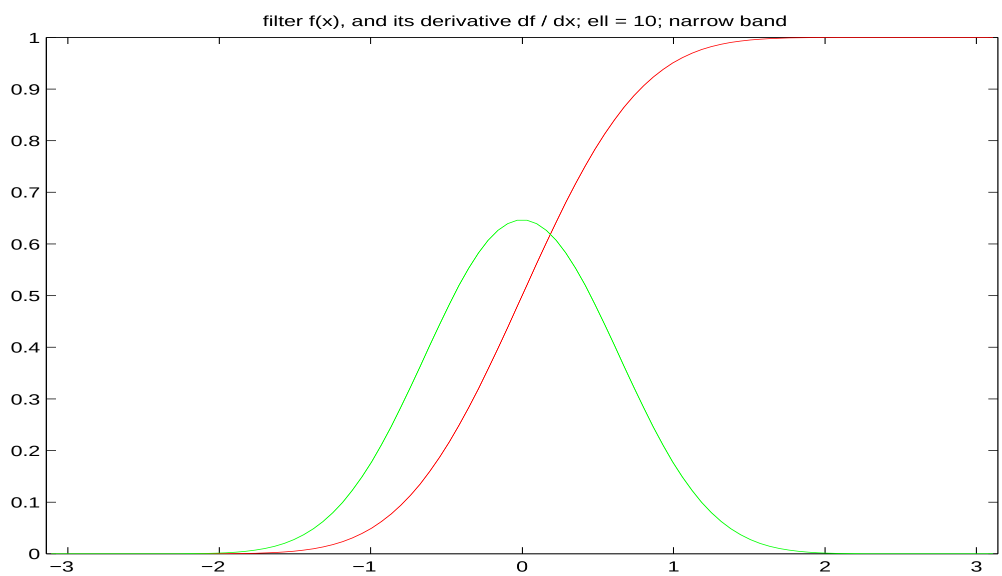

Figure 1: A typical plot of a filter f(x) and its first derivative