Midterm Project 1.2

Objective: Least-squares solution of under-determined linear system with the "physically relevant norms". Optimal design of narrow and wide bandwidth digital filters and high order quadrature formulae.

A filter f is required to be a smooth transition from 0 to 1, to be monotone ($f' \geq 0$), and to have vanishing derivatives upto certain order at the two end points $a = -\pi,\ b = \pi$. If done correctly, the function

$$f_3(x) = \frac{1}{2\pi}(x+\pi) + \sum_{j=1}^{n} c_j \sin(jx) \tag{1}$$

will satisfy the requirements (except $f' \geq 0$) with suitably chosen coefficients c_j. In the first part of the project, you have set up an m-by-n linear system for c_j, solved it with $m = n = 5$ and got a solution which gives rise to an f_3 that happens to be monotone.

There is no reason that we must choose $m = n$ and get a square linear system to which the solution is unique. On the contrary, we prefer $m < n$ so that the linear system is under-determined: there are more degrees of freedom than the number of equations. When this happens, the solution is not unique, and therefore there is freedom to choose a solution with desirable properties

Question 1. Let $D \in \mathbb{R}^{n\times n}$ be invertible. Describe with no more than 40 words (plus necessary formulae) a procedure which uses SVD to solve our problem $A \cdot c = b$ and simultaneously minimizes the 2-norm of $x = D \cdot c$. Hint: Consider the equation $A \cdot D^{-1} \cdot D \cdot c = b$. Note: for this problem to make practical sense, it is assumed that $m < n$ (what happens if $m = n$).

Question 2. Find out D if we want to solve our problem $A \cdot c = b$ and minimize the 2-norm of f_3', go to the end of this handout for details on the 2-norm of a constinuous function.

Question 3. For $m = 5,\ n = 8$, solve our problem $A \cdot c = b$ and minimize the 2-norm of $f_3^{(2m+2)}$. Show the condition number. Plot f_3 so constructed with 32 equispaced points in $[-\pi,\ \pi]$.

Question 4. For $m = 5,\ n = 31,\ \mu = 0.01$, solve our problem $A \cdot c = b$ and minimize $\|f_3'\|_2^2 + \mu\|f_3^{(4)}\|_2^2$. Show the condition number. Plot f_3 so constructed with 80 equispaced points in $[-\pi,\ \pi]$.

Remark. The above 4 problems are due Mar. 8. The next problem is due after the Spring Break. Learn the basics of Matlab Symbolic Toolbox by reading Pages 380–410 of Lindfield's text, and consulting Matlab help pages at
http://www.mathworks.com/access/helpdesk/help/toolbox/symbolic/ch1.shtml
http://www.mathworks.com/access/helpdesk/help/toolbox/symbolic/ch13.shtml
http://www.mathworks.com/access/helpdesk/help/toolbox/symbolic/ch2reftb.shtml

Question 5. For $m = 10,\ n = 23,\ \mu = 10^{-3}$, modify your code so that it runs the vpa operations with digits(50) to solve our problem $A \cdot c = b$ and minimize $\|f_3'\|_2^2 + \mu\|f_3^{(5)}\|_2^2$. Show the condition number. Plot f_3 so constructed with $k = 80$ points $x_j = -\pi + (j - 1/2)h,\ h = 2\pi/k$ in $(-\pi,\ \pi)$. Print out in double precision values of f_3 and f_3' at $x_1, x_2, \ldots, x_n$.

Remark. An observation useful for debugging: the values of f_3 and f_3' should be positive and very small near $x = -\pi$. If they are not, it means that there is a bug and it's likely that vpa operations are not implemented correctly or fully with 50 digits. This calculation is relatively cpu time intensive so first experiment with smaller m and n and verify your result by comparing it with what you've done with double precision arithmetic.

The 2-norm of a function f in $[a, b]$ is defined by the formula

$$\|f\|_2 = \left(\frac{1}{b-a} \int_{a,b} f^2(x) dx \right)^{\frac{1}{2}}; \tag{2}$$

therefore it can verified that a sine or cosine series

$$f(x) = \sum_{j=1}^{n} a_j \sin(jx), \quad \text{or} \quad f(x) = \frac{1}{2} a_0 + \sum_{j=1}^{n} a_j \cos(jx) \tag{3}$$

in $[a, b] = [-\pi, \ \pi]$ will have the 2-norm (up to a factor of 2)

$$\|f\|_2 = \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}, \quad \text{or} \quad \|f\|_2 = \left(\sum_{j=0}^{n} |a_j|^2 \right)^{\frac{1}{2}} \tag{4}$$